GR 4051 Lecture Notes: Epiphany 2020

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Contents

1 Geometric Preliminaries

This is a review of some simple geometric preliminaries that were established during the first half of the course.

1.1 Conventions

Some important words on conventions: Greek indices μ, ν run over all 4 coordinates. Latin indices i, j typically run only over spatial coordinates (x, y, z). Thus for example if I was talking about the four-velocity of a particle I might write:

$$u^\mu = (u^t, u^i) = (u^t, u^x, u^y, u^z) \tag{1.1}$$

The Minkowski metric is

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1) \ . \tag{1.2}$$

We will always use this (-,+,+,+) signature in this course, which agrees with most (but not all) GR textbooks. A useful reference is the very last page of *Einstein Gravity in a Nutshell* by A. Zee, where he summarizes a lot of textbooks and their sign conventions.

1.2 Vectors and their derivatives

What is a vector?

It is not just four numbers – if we assemble the air pressure, temperature, etc. in to a 4-component object, that will not be a vector. Instead, the basic idea here is that a vector is something that has a *real geometric meaning*, and the meaning is independent of the coordinates that we use to describe it. For this to be true, a vector must transform like a vector, i.e. if we have two coordinate systems $x^{\bar{\mu}}$ and x^μ describing the same manifold, then the relation between the components of the vector in each coordinate system is

$$V^{\bar{\mu}} = \frac{\partial x^{\bar{\mu}}}{\partial x^\mu} V^\mu \tag{1.3}$$

Note that the relation for the *down* components of the vector is

$$V_{\bar{\mu}} = \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} V_\mu \tag{1.4}$$

The Jacobian matrix appearing in the second equation is the inverse of that in the first equation. A tensor is a vector with more indices ("A tensor is something that transforms like a tensor").

Let us now consider taking the derivative of a vector. Note that the object:

$$\partial_\mu V_\sigma \tag{1.5}$$

is *not* a tensor. So this did not work. Instead we need to take a *covariant derivative*, defined as

$$\boxed{\nabla_\mu V^\sigma = \partial_\mu V^\sigma + \Gamma^\sigma_{\mu\rho} V^\rho} \tag{1.6}$$

The objects $\Gamma^{\sigma}_{\mu\rho}$ are called *connection coefficients*. They are not tensors, but their transformation properties are such that the combination above *is* a tensor. If the connection is *torsion-free* (which is pretty much always the case in physical applications) and metric-compatible (i.e. $\nabla_\mu g_{\alpha\beta} = 0$) then they are further called *Christoffel symbols* and are given by a formula

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right) \ . \tag{1.7}$$

Now it further turns out that the covariant derivative of a lower-index vector (also called a covector, or a covariant vector, or a one-form) is

$$\boxed{\nabla_\mu \omega_\sigma = \partial_\mu \omega_\sigma - \Gamma^{\rho}_{\mu\sigma} \omega_\rho} \tag{1.8}$$

Using these two properties we can take covariant derivatives of tensors with arbitrary numbers of up or down indices.

1.3 Curvature and the Riemann Tensor

Now we know that some spaces are curved and some spaces aren't. One measure (the best measure) of this curvature is whether you can parallel-transport a vector in a little curved loop and come back to the same vector or not. It should make some intuitive sense that this parallel transport is given by the commutator of two covariant derivatives. We may now directly compute:

$$[\nabla_\mu, \nabla_\nu] V^\lambda = R^{\lambda}{}_{\rho\mu\nu} V^\rho \tag{1.9}$$

where the *Riemann tensor* $R^{\lambda}{}_{\rho\mu\nu}$ is defined as

$$\boxed{R^{\lambda}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma} \Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma} \Gamma^{\sigma}_{\mu\rho}} \tag{1.10}$$

Beware signs; some books use different conventions. The Riemann tensor is a measure of how curvy a manifold is.

Here are some important properties, which are most easily checked in Riemann normal coordinates (also called *locally inertial coordinates*) in which $\Gamma = 0$ at a point:

1. Antisymmetric in first and last pairs of indices:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} \qquad R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} \tag{1.11}$$

2. Invariant under interchange of first pair of indices with second:

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \tag{1.12}$$

3. Sum over cyclic permutations of last three indices vanishes:

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0. \tag{1.13}$$

   These are all equations that are all true locally at a point. There is also a differential identity that relates the value of the Riemann tensor at neighbouring points, called the

4. Bianchi identity:

$$\nabla_{[\gamma} R_{\alpha\beta]\mu\nu} = 0 \tag{1.14}$$

The Riemann tensor has a lot of indices and is a lot of information to deal with. A very tedious counting problem lets you determine that it has 20 independent components in 4 dimensions.

To get more manageable objects, we can contract in various ways. For example, it is often very useful to contract on the first and third indices to define the *Ricci* tensor:

$$\boxed{R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}}, \tag{1.15}$$

whose trace defines the unimaginatively named Ricci *scalar*

$$\boxed{R = g^{\mu\nu} R_{\mu\nu}}\ . \tag{1.16}$$

If we now contract indices twice on the Bianchi identity (1.14) we find the *contracted Bianchi identity*.

$$\nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R \tag{1.17}$$

Given the contracted Bianchi identity, we see that there is a particular two index tensor that does two things: (a) it depends on the curvature of the manifold, and (b) is is *divergenceless*. This tensor is fairly important in what follows, and is called the *Einstein tensor*:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{1.18}$$

Given (1.17) it is immediately true that $\nabla_{\mu} G^{\mu\nu} = 0$. This will be important.

2 Gravity from Curvature

Having already established all possible geometric preliminaries, we are ready to plunge into a study of the physical implications of general relativity, i.e. to see how *gravity* actually emerges from this setup. Before doing this in full general relativistic glory its good to remind ourselves how this works in conventional *non*-relativistic Newtonian physics.

2.1 Newtonian physics

For a bit, let's forget all this fancy stuff about curved space etc. and let's recall how gravity works in elementary physics. In conventional Newtonian mechanics, we have a field filling all of space called the Newtonian gravitational potential $\Phi(x)$. (Presumably) immediately after the apple fell on his head, Newton realized that all matter contributes to this Newtonian potential via an equation that we now write as Poisson's equation:

$$\nabla^2 \Phi(x) = \delta^{ij} \partial_i \partial_j \Phi(x) = 4\pi G \rho(x) \tag{2.1}$$

Given a point mass source $\rho(x) = m\delta^{(3)}(x)$, this results in a gravitational potential that falls off as

$$\Phi(x) = -\frac{GM}{r} \tag{2.2}$$

as you can verify in Problem 2 of this week's homework. Now if you have a point particle of mass m, it feels a gravitational force that is given by

$$F^i_{\text{grav}} = -m \partial_i \Phi(x) \tag{2.3}$$

And thus Newton's second law $F = ma$ tells us that the acceleration is related to this potential by:

$$m \frac{d^2 x^i}{dt^2} = -m \partial_i \Phi \tag{2.4}$$

where the m's now cancel. Note that from Newtonian physics this appears as a bit of a coincidence, but in general relativity this is a manifestation of the *equivalence principle* and is required. Also note that I am being cavalier about the indices here, because I am in flat space and it is okay. In the next section I will be careful again.

We know experimentally that these equations work quite well for slowly moving objects and weak gravitational fields; thus we must be able to reproduce them from Einstein gravity.

2.2 Geodesics and the weak field limit

We now will perform a similar analysis in general relativity. Recall the notation: μ runs over all 4 coordinates, i, j will run over only space.

To understand how these things work, we should remind ourselves that the *equivalence principle* states that *physics on sufficiently small scales behaves as though we are in flat space.* In other words, to figure out what the right equations are, we should just ask ourselves what they would be in flat space, write them in a tensorial way, and then *assert* that the resulting equations are correct in curved space.

Consider being in empty space, far from everything you can imagine. In that case, from elementary physics we know that in flat space, particles move in straight lines. In Cartesian coordinates in flat space the right equation is:

$$\frac{d^2x^\mu}{ds^2} = 0 \tag{2.5}$$

However this equation is not actually coordinate invariant. As you all know, its covariant version is the geodesic equation. This is

$$\frac{D^2x^\mu}{ds^2} = 0 \qquad \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{2.6}$$

This equation should somehow reduce to (2.4) in the non-relativistic weak field limit. To see how this can happen, let's first put down Cartesian coordinates (t, x, y, z). Now weak-field means

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1 \tag{2.7}$$

From now on will work only to first order in $h_{\mu\nu}$. Non-relativistic means that we move mostly in time, not in space, so

$$\frac{dx^i}{ds} \ll 1 \tag{2.8}$$

We will also work to the lowest non-trivial order in $\frac{dx^i}{ds}$. As described in the first part of the course, we normally pick the parameter along the path s to be the proper time (or length), because this simplifies the equations. (This is called *affine parametrization*). In that case we have that the four-velocity is normalized to 1. We thus find

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -1 \qquad \to \qquad \left(\frac{dt}{ds}\right)^2 - \left(\frac{dx^i}{ds}\frac{dx^j}{ds}\right)\delta_{ij} = 1 \tag{2.9}$$

where we assumed that $g_{\mu\nu} \approx \eta_{\mu\nu}$. But as we will work to lowest non-trivial order in $\frac{dx^i}{ds}$, we may omit the second term and conclude:

$$\frac{dt}{ds} \approx 1 \tag{2.10}$$

That means the only fun equation above is

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{tt}\frac{dt}{ds}\frac{dt}{ds} = 0 \tag{2.11}$$

Now we need to compute the Christoffel symbol in the weak-field limit. Let's also take the metric to be time-independent, $\partial_t h_{ij} = 0$. We only need one Christoffel:

$$\Gamma^i_{tt} = \frac{1}{2} g^{i\mu} \left(\partial_t g_{\mu t} + \partial_t g_{\mu t} - \partial_\mu g_{tt} \right) \approx -\frac{1}{2} \eta^{ij} \partial_j h_{tt} \tag{2.12}$$

Using the fact that $\frac{dt}{ds} \approx 1$, the first equation becomes

$$\frac{d^2 x^i}{ds^2} = \frac{1}{2} \partial_i h_{tt} \tag{2.13}$$

Note there is no difference between s and t; thus, this looks like Newton's equation! All we need is to define

$$h_{tt} = -2\Phi(x) \tag{2.14}$$

with Φ the Newtonian potential. Note what this means: *particles moving in a weakly curved space with metric*

$$g_{tt} \approx -(1 + 2\Phi(x)) \tag{2.15}$$

look like they're moving in a Newtonian gravitational potential Φ.

2.3 Einstein's equation

We now need to search for the analog of the first equation (2.1): in other words, we have just learned how matter (ok, particles) behave in curved space; but we now need to learn how curved space responds to matter. This is given by *Einstein's equation.* You all know it already, but I want to motivate a bit because this helps understand why it has the structure it does. We expect it to be something of the form

$$\text{(Curviness of space)} = \text{(Something involving matter)} \tag{2.16}$$

Now the equation must describe the dynamics of 10 degrees of freedom – so we will need a matrix equation, or a two-index equation. What involves matter and has two indices? Clearly the right-hand side must be the stress tensor.

$$\text{(Curviness of space)} = T_{\mu\nu} \tag{2.17}$$

What can be on the left-hand side? We need something that has two indices. One candidate is $R_{\mu\nu}$. Why doesn't that work? The right-hand side is conserved, and so the left-hand side must also be. It turns out that the only simple candidate is the Einstein tensor defined earlier. So now we basically have

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}} \tag{2.18}$$

where I have put $8\pi G$ in the right place with the benefit of hindsight. Aren't they beautiful and simple? Depending on your taste, these have been called "the most beautiful equations of all time," etc. etc. They are kind of interesting; even though they are very simple to write down, they are actually fiendishly fiendishly complicated, as the relation between $G_{\mu\nu}$ and $g_{\mu\nu}$ is quite intricate. Very few exact solutions are known – we will discuss many of those that are in the remainder of this class.

Before doing this, we need to understand why the constant on the right hand side is $8\pi G$. We will do this by appealing to the same trick we did previously; we will work out a particular component of this equation and relate it to (2.1). It turns out this is simplest to do if we first take the trace to find that

$$-R = 8\pi G g^{\mu\nu} T_{\mu\nu} \equiv 8\pi G T \tag{2.19}$$

and then insert this back into the equation above to find

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right) \tag{2.20}$$

We will now evaluate just the tt component of this equation. We'll do it on a metric (2.7) that is very close to flat space.

First the right-hand side. Consider, as discussed in the previous half of the course, a *perfect fluid* stress tensor:

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu} \tag{2.21}$$

Here U^μ is the four-velocity of the fluid elements. Let's take the fluid to be sitting around not doing much. This means that its fluid elements move only in time and not in space. Just like before, we have

$$U^t \approx 1 \tag{2.22}$$

Let's also set $p = 0$, i.e. the matter is *dust*. For a non-relativistic fluid, this makes sense, because there are secret relative factors of c in the expression above that kill it. So now we can work out

$$T_{tt}(x) = \rho(x) \qquad T(x) = -\rho(x) \tag{2.23}$$

Next, we need to work out R_{tt}. Ok, so we need $R_{tt} = R^j{}_{tjt}$. Let's now compute

$$R^i{}_{tjt} = \partial_j \Gamma^i_{tt} - \partial_t \Gamma^i_{jt} + \mathcal{O}(\Gamma^2) \tag{2.24}$$

We can ignore the last few terms because we are working only to first order in h and $\Gamma \sim \mathcal{O}(h)$; and we can ignore the second term because we assume everything is static. So we find:

$$R_{tt} = R^j{}_{tjt} = \partial_j \Gamma^j_{tt} = -\frac{1}{2}\delta^{ij}\partial_i\partial_j h_{tt} \tag{2.25}$$

where in the second equality we used the fact from last lecture that $\Gamma^j_{tt} = -\frac{1}{2}\delta^{ij}\partial_i h_{tt}$. Putting all of these pieces together we find that

$$-\frac{1}{2}\delta^{ij}\partial_i\partial_j h_{tt} = 8\pi G\left(T_{tt} - \frac{T}{2}g_{tt}\right) = 4\pi G\rho(x) \tag{2.26}$$

Now recall from before that we know that the relation between the Newtonian potential and the metric perturbation is $h_{tt} = -2\Phi(x)$. That means that this equation is

$$\nabla^2\Phi = 4\pi G\rho(x) \tag{2.27}$$

This is precisely what we expected from ordinary physics, (i.e. (2.1)). Note if we had put some other factor there it would not have worked out and we would not have recovered ordinary non-relativistic physics in the limit.

Thus we correctly recover Newton in the limit. Of course we do a great deal more, as we will see in the rest of the course.

2.4 The Einstein-Hilbert Action

Now we are going to learn how to derive these equations from the stationary points of an action. This is very useful; actions are usually the most elegant way to formulate classical physics, and they make the transition to quantum mechanics easier (though, sadly, we will not discuss what happens when you do that to gravity in this course.)

2.4.1 Quick review of calculus of variations

So, what is the action? Recall that the action for a point particle is something like

$$S[x] = \int ds L(x, \dot{x}) = \int ds \left(\frac{1}{2}\dot{x}^2 - V(x)\right) \tag{2.28}$$

It is a functional of the particle trajectory $x(s)$. and we demand that the action be stationary with respect to small variations of the particle trajectory δx to find the equations of motion. Let's just remind ourselves very quickly how this works:

$$\delta S[x] = \int ds \left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}\right) \tag{2.29}$$

Now we integrate by parts on the last term to find:

$$\delta S[x] = \int ds \left(\frac{\partial L}{\partial x}\delta x(s) - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x(s)\right) \tag{2.30}$$

So the condition for stationarity is the familiar Euler-Lagrange equations:

$$\frac{\partial L}{\partial x} - \frac{d}{ds}\frac{\partial L}{\partial \dot{x}} = 0 \tag{2.31}$$

Let me take a moment to define the *functional derivative* $S[x]$ with respect to $\delta x(s)$: it is defined as the thing multiplying $\delta x(s)$ in the expression above, i.e. in this case we have:

$$\frac{\delta S[x]}{\delta x(s)} = \left(\frac{\partial L}{\partial x} - \frac{d}{ds}\frac{\partial L}{\partial \dot{x}}\right)\Bigg|_s \tag{2.32}$$

2.4.2 Varying the Einstein-Hilbert action

Now we will do the same for gravity, i.e. the dynamics of the metric $g_{\mu\nu}(x)$. The first thing to note is that the metric is a field defined over all space and time; thus the action is an integral over all of this, i.e.

$$S[g] = \int dt d^3x\,(\text{something}) = \int d^4x\,(\text{something}) \tag{2.33}$$

Now the action determines physics, and physics is coordinate-invariant. Thus we want the numerical value of the action to also be coordinate-invariant, i.e. a scalar. Let's think for a second about what this means. First of all, is the integration measure d^4x invariant? *No.* By well-known formulas of calculus, you know that under a coordinate transformation the integration measure changes by a factor of the determinant of the Jacobian:

$$d^4\bar{x} = d^4x \det\left(\frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}}\right) \tag{2.34}$$

(recall $dxdy = rdrd\theta$ etc. etc.) How do we deal with this? It is quite simple: it turns out that the determinant of the metric transforms as

$$\det\left(g_{\bar{\mu}\bar{\nu}}\right) = \det\left(\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}} g_{\mu\nu} \frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}\right) = \det\left(\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\right)^2 \det\left(g_{\mu\nu}\right) \tag{2.35}$$

Now the Jacobian matrix appearing in (2.35) is precisely the inverse of that appearing in (2.34); thus the combination

$$d^4\bar{x}\sqrt{-\det\left(g_{\bar{\mu}\bar{\nu}}\right)} = d^4x\sqrt{-\det\left(g_{\mu\nu}\right)} \tag{2.36}$$

is indeed invariant. (This minus sign is because g has 3 negative eigenvalues and 1 positive.) So this is then the correct measure for integrating anything. We will abbreviate it as

$$d^4x\sqrt{-g} \qquad g \equiv \det\left(g_{\mu\nu}\right) \tag{2.37}$$

We can now refine the action integral to be

$$S[g] = \int d^4x\sqrt{-g}\,(\text{something scalar}) \tag{2.38}$$

Now the action should be a scalar thing that involves derivatives of the metric. It turns out that really the only reasonable choice is the Ricci scalar. This was first realized by Hilbert, and the action is thus called the *Einstein-Hilbert action*:

$$\boxed{S_{EH}[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R} \tag{2.39}$$

So we now need to vary the action with respect to $g_{\mu\nu}$. In fact it is easier to vary it with respect to the inverse metric $\delta g^{\mu\nu}$. Note that the variation of the inverse metric and the metric are related by a *minus* sign: for any matrix M, consider

$$\delta M^{-1} = -M^{-1}\delta M M^{-1} \tag{2.40}$$

which for the case of the metric means

$$\delta g^{\mu\nu} = -g^{\mu\rho}\delta g_{\rho\sigma}g^{\nu\sigma} \tag{2.41}$$

Now we turn to the variation. This basically has three parts.

$$\delta S = \frac{1}{16\pi G}\int d^4x\left((\delta\sqrt{-g})R + \sqrt{-g}\left[(\delta g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right]\right) \tag{2.42}$$

Let's call the terms $\delta S_{1,2,3}$. $\delta S_{1,2}$ aren't bad; δS_3 looks more complicated, as we need to work out the variation of the Riemann tensor. This is not quite so fiendish as it appears. We will work out the variation only in terms of the Christoffel symbols, i.e. from its definition

$$R^{\lambda}{}_{\mu\alpha\nu} = \partial_\alpha\Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\alpha\sigma}\Gamma^\sigma_{\nu\mu} - (\alpha\leftrightarrow\nu) \tag{2.43}$$

we vary in terms of $\delta\Gamma^\alpha_{\beta\alpha}$ to find

$$\delta R^{\lambda}{}_{\mu\alpha\nu} = \partial_\alpha\delta\Gamma^\lambda_{\nu\mu} + \delta\Gamma^\lambda_{\alpha\sigma}\Gamma^\sigma_{\nu\mu} + \Gamma^\lambda_{\alpha\sigma}\delta\Gamma^\sigma_{\nu\mu} - (\alpha\leftrightarrow\nu) \tag{2.44}$$

Now this is interesting: note that even though the Christoffel symbol is *not* a tensor, its variation is the difference between two connections and thus *is* a tensor. So it makes sense to take its covariant derivative; and indeed the thing appearing there is in fact

$$\delta R^{\lambda}{}_{\mu\alpha\nu} = \nabla_\alpha\delta\Gamma^\lambda_{\nu\mu} - \nabla_\nu\delta\Gamma^\lambda_{\alpha\mu} \tag{2.45}$$

Now let us note something interesting. We see that the term that we want is now two contractions:

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\delta R^{\lambda}{}_{\mu\lambda\nu} = \nabla_\lambda v^\lambda \tag{2.46}$$

where I have defined the vector

$$v^\lambda = \delta\Gamma^\lambda_{\mu\nu}g^{\mu\nu} - g^{\lambda\alpha}\delta\Gamma^\sigma_{\sigma\alpha} \tag{2.47}$$

Thus the term δS_3 is

$$\delta S_3 = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\nabla_\lambda v^\lambda \tag{2.48}$$

However this is a total derivative! In other words, you can integrate it by parts using the curved-space version of Gauss's law to say that

$$\int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\lambda v^\lambda = \int_{\partial\mathcal{M}} d^3x\sqrt{-h}n_\alpha v^\alpha \tag{2.49}$$

where $\mathcal{M}$ is the manifold and $\partial\mathcal{M}$ is its boundary, where $h_{\mu\nu}$ is the metric on this boundary. So this is not actually a local integral that will contribute to the equations of motion; its just a boundary term that we ignore.

So now the only thing that we have left to compute is

$$\delta S_1 \propto \int d^4x (\delta\sqrt{-g})R \tag{2.50}$$

To compute the variation of the determinant of a metric, we use the fact that for any $n \times n$ matrix A

$$\det A = \exp \mathrm{Tr} \log A \tag{2.51}$$

Proof: let's diagonalize the matrix A. Call its eigenvectors λ_i. Then the left hand side is

$$\det A = \lambda_1\lambda_2\lambda_3 \cdots \lambda_n \tag{2.52}$$

The logarithm of a matrix A is defined as the matrix with the same eigenvectors as A, but with eigenvalues each equal to the logarithm of the eigenvalues of A. Thus it is clear that the right hand side is

$$\exp\left(\sum_{i=1}^{n} \log \lambda_i\right) = \lambda_1\lambda_2 \cdots \lambda_n \tag{2.53}$$

i.e. its equal to the left hand side.

This means that

$$\delta(\det A) = \exp \mathrm{Tr} \log A \delta\left(\mathrm{Tr} \log A\right) = \det A \, \mathrm{Tr}\left(A^{-1}\delta A\right) \tag{2.54}$$

Applied to the metric this means that

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} = -g g_{\mu\nu}\delta g^{\mu\nu} \tag{2.55}$$

and we can conclude that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} \tag{2.56}$$

Assembling the pieces we find that the variation of the Einstein-Hilbert action is

$$\delta S_1 + \delta S_2 = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right)\delta g^{\mu\nu} \tag{2.57}$$

This is the result we were looking for. Demanding that this variation vanish we find that

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 0, \tag{2.58}$$

i.e. the vacuum Einstein equations!

2.5 Matter

So how do we obtain the other part? This comes from the fact that typically there will be other terms in the action, i.e. there will be a "matter" bit that also depends on $g_{\mu\nu}$. The total action is

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_m[g] \tag{2.59}$$

where S_m is the action for matter degrees of freedom. We will discuss some examples shortly. But for now, let's just formally vary it to see that we find:

$$\delta S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} + \int d^4x \frac{\delta S_m}{\delta g^{\mu\nu}}\delta g^{\mu\nu} \tag{2.60}$$

This variation will be stationary on solutions to Einstein's equations as we know and love them (2.18) if we simply state that the stress-energy tensor $T_{\mu\nu}$ is defined to be:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad \boxed{T_{\mu\nu}(x) = -\frac{2}{\sqrt{-g}}\frac{\delta S_m[g]}{\delta g^{\mu\nu}(x)}} \tag{2.61}$$

This expression involves a "functional derivative"; as we discussed around (2.32), it really just means that you take the variation and look at the part multiplying $\delta g^{\mu\nu}$.

Let us now look at some examples:

2.5.1 Cosmological constant

The very simplest term that we can add to the action is a cosmological constant, i.e. add this term to the action as

$$S_\Lambda = -\frac{1}{8\pi G}\int d^4x\sqrt{-g}\Lambda \tag{2.62}$$

Its variation comes entirely from the determinant of the metric:

$$\delta S_\Lambda = -\frac{1}{8\pi G}\int d^4x(\delta\sqrt{-g})\Lambda = +\frac{1}{16\pi G}\int d^4x\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\Lambda \tag{2.63}$$

So from here we see that it contributes a term to the stress energy tensor that is

$$\boxed{T^{(\Lambda)}_{\mu\nu} = -\frac{1}{8\pi G}\Lambda g_{\mu\nu}} \tag{2.64}$$

We will come back to this in the section on cosmology.

2.5.2 Electromagnetism

In real life we also have *electromagnetism.* Everyone here should be familiar with at least the *effects* of electromagnetism – it results in such diverse phenomena as light, lightning, static electricity, television, etc. In this choice of topics I am being a little bit facetious here; in fact almost all observable phenomena at everyday scales rely crucially on EM.

Let us take a second to develop electromagnetism in a sophisticated formalism. The basic degree of freedom here is the *vector potential* $A_\mu(x)$; this is a 4-(co)vector field that depends on space and time. We also usually construct the *field strength tensor*. This is defined as

$$F_{\mu\nu}(x) = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{2.65}$$

I want to take a second to point out that actually this particular expression can also be written in terms of partials. To be more precise, let's expand out each covariant derivative using the Christoffel symbols:

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \Gamma^\sigma_{\mu\nu}A_\sigma - \partial_\nu A_\mu + \Gamma^\sigma_{\nu\mu}A_\sigma \tag{2.66}$$

Now because the Christoffels are symmetric, we have $\Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}$; thus the two Christoffel symbols *cancel*, and we can write

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.67}$$

Thus the expression for F takes the same form whether we use partial or covariant derivatives. This is a very special case; generally this doesn't happen. (In this case it has to do with the existence of *differential forms*, which I will not discuss any further in this course).

Now whenever we want to specify a theory, you should write down its action as a function of the basic degree of freedom. In this case the action functional also needs the metric, and it turns out to be:

$$S_{EM}[A,g] = -\frac{1}{16\pi}\int d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{2.68}$$

The theory described by this action is also called *Maxwell electrodynamics.*

What are the equations of motion of the theory? To derive these, we vary the action with respect to A_μ:

$$\delta_A S_{EM}[A,g] = -\frac{2}{16\pi}\int d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}\delta F_{\rho\sigma} = -\frac{2}{16\pi}\int d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}\left(\nabla_\rho\delta A_\sigma - \nabla_\sigma\delta A_\rho\right) \tag{2.69}$$

$$= +\frac{4}{16\pi}\int d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}(\nabla_\rho F_{\mu\nu})\delta A_\sigma \tag{2.70}$$

Thus the equation of motion is

$$\nabla_\mu F^{\mu\nu} = 0 \tag{2.71}$$

These are Maxwell's equations (with no charges present) written in relativistic notation. Those of you who have seen Maxwell's equation in other courses may find them a bit unfamiliar; where are the electric and magnetic fields? If you go to flat space and pick a rest frame (i.e. pick a particular choice of time coordinate t), then they are:

$$F^{ti} = E^i \qquad F^{ij} = \epsilon^{ijk}B_k \tag{2.72}$$

It is then a soothing exercise that working out (2.71) in components you get the form of Maxwell's equations that are on those annoying t-shirts.

Leaving aside Maxwell's equations, let us now compute the stress energy tensor for the Maxwell field. We compute the variation with respect to g rather than A to find

$$\delta_g S_{EM}[A,g] = -\frac{1}{16\pi}\int d^4x\left(\sqrt{-g}\left(\delta g^{\mu\rho}g^{\nu\sigma} + g^{\mu\rho}\delta g^{\nu\sigma}\right)F_{\mu\nu}F_{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\delta\sqrt{-g}\right) \tag{2.73}$$

$$= -\frac{1}{16\pi}\int d^4x\sqrt{-g}\left(2g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\rho}\right)\delta g^{\mu\rho} \tag{2.74}$$

(Note that you might have been worried that the covariant derivatives in the definition of F (2.65) also depend on g; however we can write F in terms of partial derivatives if we like, as in (2.67). So there is no dependence on g there).

So from here we can easily read off the stress-energy tensor:

$$\boxed{T^{(EM)}_{\mu\rho} = \frac{1}{4\pi}\left(g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\rho}\right)} \tag{2.75}$$

This is familiar from earlier in the course; it tells you how electromagnetism affects gravity.

2.5.3 A moving particle

I will mention only one other example. Consider a point particle moving through space. We know how gravity affects it, but how does *it* affect gravity? It has a mass and energy, so it should affect gravity (a little bit!) What is its stress tensor? If we knew the action, we could find the stress-energy tensor.

Plot.

We know this already: the action of a point particle of mass m is actually proportional to its *proper time*:

$$S[X] = m\int ds\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}} \tag{2.76}$$

The prefactor is the mass of the particle. Here s is a parameter along the worldline.

Note by the way that the action is invariant under reparametrizations: i.e imagine picking a different coordinate s' to parametrize the worldline, where s' is a function of s: then we have

$$S[X] = m\int ds'\frac{ds}{ds'}\sqrt{-g_{\mu\nu}(x)\frac{ds'}{ds}\frac{dx^\mu}{ds'}\frac{ds'}{ds}\frac{dx^\nu}{ds'}} = m\int ds'\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{ds'}\frac{dx^\nu}{ds'}}, \tag{2.77}$$

i.e. the action takes exactly the same form as a function of the new parameter s'. Note also that computing its variation with respect to x will just result in the geodesic equation, as you know from the first term.

Now again we can compute the stress-energy of the particle by varying with respect to $g_{\mu\nu}$. This involves delta functions and will eventually be a homework problem.

2.5.4 A little philosophy

We know that everything *feels* gravity; in a slightly crude form the equivalence principle states that everything falls at the same rate, and that means that there is no way to shield something from the effects of the gravitational field. With our new sophisticated understanding, we understand that this is because freely falling objects follow geodesics on the metric that defines spacetime.

However it is interesting to note that from our discussion of the Einstein-Hilbert action, we see that not only does everything feel gravity, gravity also feels everything! Basically[1] any term you can write in an action will involve the metric to tie together indices, but we see that this invariably means that it will contribute to the stress energy tensor. Thus anything that cares about how big space is or how quickly time ticks will itself leave its imprint on space and time.

[1] There is something called a topological field theory that is an exception to this; these are odd objects that do not really have have any degrees of freedom, and it is a bit philosophical whether or not you want to call them a “thing”.

3 The Schwarzschild Solution

Now that we know Einstein's equation, let's try to solve it. In general this is very difficult, and there are very few exact solutions known. So we will first try to solve the equations in *vacuum*, i.e. with $T_{\mu\nu} = 0$, for a spherically symmetric and *static* mass distribution, e.g. outside a star. This will result in a lot of fun, ending in black holes.

3.0.1 A cute but ultimately wrong calculation

However there is a calculation involved first. As it is a little bit tedious, let's give ourselves a little treat from Laplace (in 1796!). Suppose you need to convince a first-year (or a biologist or well-read art historian or someone else who has studied only Newtonian physics...) that black holes exist. You might say the following: in ordinary Newtonian gravity, what is the *escape velocity* of a particle that is a distance R from a mass M? Recall this is done by energy conservation: the initial situation is a particle in a gravitational potential well with kinetic $\frac{1}{2}mv^2$ and potential energy $-\frac{GMm}{R}$, and the final situation is a particle at rest at infinity (i.e. net energy 0). Thus the equation of interest is

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = 0 \qquad \to \qquad v^2 = \frac{2GM}{R} \tag{3.1}$$

Note: as you make R smaller and smaller, you need a higher and higher velocity to escape. Eventually this will hit the speed of light c! This happens at the radius

$$R = \frac{2GM}{c^2} \tag{3.2}$$

This suggests that something weird is happening here – apparently if we could compress the sun into a ball that is sufficiently small, light itself would not be able to escape from the surface due to its gravitational pull! This *suggests* that something fun might happen at that radius. Note that in reality this calculation is *wrong*, as Newtonian gravity doesn't apply to things moving close to the speed of light. The true situation is much more fun.

3.1 Deriving the solution

Being motivated by this, let's now treat the problem correctly using the full machinery of GR. The equations are simply

$$R_{\mu\nu} = 0 \tag{3.3}$$

Now we solve them. First we need to write down a metric *ansatz*. What does it mean to be *static*? This has a technical meaning that we will discuss later, but for now we will just say that it means that the spacetime *has* a time coordinate t and that nothing depends on t and also that there are no cross terms like $dtdx^i$ in the metric.

Next, what does spherically symmetric mean? Again, there is a technical meaning, but recall flat space written in polar coordinates for a second:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{3.4}$$

Inspired by this, we will take *spherically symmetric* to mean that the spacetime *has* coordinates (θ, ϕ) and that the dependence of the metric on these coordinates is via the familiar $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. If we were to alter this it would correspond to squashing the sphere [2].

[2] The technical meaning of spherically symmetric is: *there exists a set of Killing vectors whose Lie brackets form the Lie algebra of $SO(3)$.*

So in that case, the most general spherically symmetric static metric with the same signature as flat space takes the form

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + e^{2C(r)}r^2 d\Omega^2 \tag{3.5}$$

The exponential form makes certain things a bit simpler: roughly speaking it seems to maintain the signature of the metric and keeps dt as a time coordinate[3]. Now we *can* actually simplify this a bit. Consider defining a new radial coordinate in the following way:

$$e^{C(r)}r \equiv \bar{r} \qquad d\bar{r} = e^{C(r)}(1 + rC'(r))dr \tag{3.6}$$

This changes the last term to $\bar{r}^2 d\Omega^2$. In terms of the new coordinate we thus have

$$ds^2 = e^{2A(\bar{r})}dt^2 - e^{2B(\bar{r})-2C(\bar{r})}(1 + rC'(r))^{-2}d\bar{r}^2 - \bar{r}^2 d\Omega^2 \tag{3.7}$$

Now these are all just *labels*: so lets redefine:

$$\bar{r} \to r \qquad e^{2B(\bar{r})-2C(\bar{r})}(1 + rC'(r))^{-2} \to e^{2B(r)} \tag{3.8}$$

After this series of manipulations we find that the metric becomes

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2 d\Omega^2 \tag{3.9}$$

Which is exactly the same as (3.5), except that the final $e^{2C(r)}$ has vanished. Physically speaking, we have decided to *pick* the radial coordinate r so that the proper area of a constant-r 2-sphere is always $4\pi r^2$. This is nice, because it gives us one less function to worry about. Note there is something mildly deep going on here – in general relativity, the coordinates themselves have no meaning, and it is only the relationship between coordinates and metric components that have physical meaning.

This is as far as we can go without doing any work. Now we actually need to solve the equations. So we should first compute the Christoffels, then the Riemann tensor, then the Ricci tensor. I encourage everyone to go home and do this. It is a character-building exercise when done in private.

After some algebra we find:

$$R_{tt} = e^{2(A-B)}\left(A'^2 + \frac{2A'}{r} - A'B' + A''\right) \tag{3.10}$$

$$R_{rr} = -A'^2 + \frac{2B'}{r} + A'B' - A'' \tag{3.11}$$

$$R_{\theta\theta} = e^{-2B}(-1 + e^{2B} - rA' + rB') \tag{3.12}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \tag{3.13}$$

$R_{\phi\phi}$ is $\sin^2\theta$ times $R_{\theta\theta}$ because of the assumption of spherical symmetry. We now need to set all of these to zero.

The first thing we do is consider the equation

$$e^{-2(A-B)}R_{tt} + R_{rr} = \frac{2}{r}(A' + B') = 0 \tag{3.14}$$

This means that $A(r) + B(r) = c$. Now note that by rescaling the time coordinate $t \to te^{-2c}$ we can set the constant c to 0 with no loss of generality. Thus we have $B(r) = -A(r)$.

Next, we plug this into the equation for $R_{\theta\theta}$. We find

$$-1 + e^{-2A} - 2rA' = 0 \tag{3.15}$$

$$\tag{3.16}$$

[3] We'll see later that actually it doesn't really do a terribly good job of this.

multiply through by e^{2A} and rearrange to get

$$e^{2A}(2rA' + 1) = 1 \tag{3.17}$$

$$\partial_r(re^{2A(r)}) = 1 \tag{3.18}$$

This is easy to integrate! We find

$$re^{2A(r)} = r - r_s \tag{3.19}$$

where r_s is an integration constant called the Schwarzschild radius. This is it! We now know $e^{2A(r)}$. We can now plug this into the original metric ansatz to find the celebrated Schwarzschild metric!

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3.20}$$

Before moving on, did we actually solve all the equations? We solved $R_{\theta\theta} = 0$ and thus also $R_{\phi\phi} = 0$. We solved one particular linear combination of R_{tt} and R_{rr}; as there are 2 independent equations, to be complete we also need to solve another linearly independent combination, which we can just take to be (3.10). Since we have no freedom left, this had better work: indeed plugging in the solution and going through the algebra, we see that it does.

Mathematically, we are done! Physically, we still need to interpret the meaning of r_s. Note that if $r_s = 0$, we just have flat space. Let's now consider taking $r \gg r_s$. In that case the metric is very close to flat space. Now we know that if a metric is very close to flat space, we can interpret the deviation of g_{tt} from 1 as being the Newtonian potential:

$$g_{tt} \approx -(1 + 2\Phi(x)) \tag{3.21}$$

Now, far outside a mass distribution of mass M, we know that the ordinary Newtonian potential is $\Phi(r) = -\frac{GM}{r}$. Thus this matches nicely, provided we identify

$$r_s = 2GM \tag{3.22}$$

Thus we find the Schwarzschild metric, in all its glory:

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2} \tag{3.23}$$

This is a very simple and profound result, and is one of the most important known solutions to general relativity. It is equivalent to the Coulomb potential in electrodynamics.

Note that the Schwarzschild radius is actually the *same thing* that we got from Laplace at the beginning of lecture! (Note also that I didn't set c to 1 in that section). As best as I know, this exact equality is a coincidence, as Laplace did not treat relativistic physics correctly. It looks like something bad is happening there; indeed, as we will see, this is actually the *event horizon* of the black hole. It is less bad than it looks.

For the sun, if you plug in the numbers, you find that

$$r_{\text{sun}} = \frac{2GM_{\text{sun}}}{c^2} \approx 3 \text{ km} \tag{3.24}$$

for this. This is far smaller than the radius of the sun itself (which is about 700,000 km); recall that our derivation only works in the vacuum region outside the sun, so it doesn't work in the interior. Thus we need not worry about this dangerous radius for the sun or for anything in our solar system, which is what we sill discuss next.

3.2 Geodesic motion in the Schwarzschild metric

This explains the gravitational field produced by a large massive object; we now want to understand how test particles move around it. For example, the sun creates a gravitational field, and we want to see how the earth (or, as we will see, more interestingly, Mercury) move around it. You can play with an online web game on my website that lets you fly a spaceship around a black hole here.

3.2.1 Killing vectors and symmetries

First, let's remind ourselves how Killing vectors work. Recall the geodesic equation: let's let the geodesic be a path $x^\mu(s)$, where the four-velocity is

$$u^\mu = \frac{dx^\mu}{ds} \tag{3.25}$$

Recall from the first term that a geodesic (see e.g. page 12 of first term lecture notes) satisfies

$$u^\mu \nabla_\mu u^\nu = 0 \ . \tag{3.26}$$

Now there is something called a *Killing vector*, which I now remind you of: recall from previous lectures that a Killing vector satisfies *Killing's equation:*

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \ . \tag{3.27}$$

If you write it out in coordinates, then it turns out that if the metric is independent of a coordinate x^0 then the vector field $\frac{\partial}{\partial x^0}$ is a Killing vector, i.e. $\xi^\mu(x) = \delta^\mu_0$.

This is useful because once you identify the Killing vectors, the quantity

$$Q_\xi \equiv \xi_\mu u^\mu \tag{3.28}$$

is a conserved quantity, which turns out to be constant along a geodesic $x^\mu(s)$:

$$\begin{aligned} \frac{d}{ds} Q_\xi(s) = \frac{dx^\mu}{ds} \nabla_\mu Q_\xi(s) &= u^\mu \nabla_\mu (\xi_\rho u^\rho) && (3.29)\\ &= u^\mu \left((\nabla_\mu \xi_\rho) u_\rho + \xi_\rho \nabla_\mu u^\rho \right) && (3.30)\\ &= u^\mu \left(u^\rho \frac{1}{2} \left(\nabla_\mu \xi_\rho - \nabla_\rho \xi_\mu \right) + 0 \right) && (3.31)\\ &= 0 && (3.32) \end{aligned}$$

where in the last equality I used Killing's equation (3.27) on the first term and the geodesic equation (**??**) on the last term. The upshot is that whenever the metric doesn't depend on a coordinate, we can identify a conserved quantity.

I will point out that this is an example of a much deeper phenomenon. The fact that the metric doesn't depend on (say) x^0 means that $g_{\mu\nu}(x_0 + a) = g_{\mu\nu}(x_0)$: in other words, there is a symmetry under the operation

$$x^0 \to x^0 + a \tag{3.33}$$

i.e. *translations in* x_0. In physics, symmetry almost always leads to a conserved quantity such as Q_ξ above: the precise connection is something called Noether's theorem, which you may remember from courses on classical mechanics. Killing vectors may be thought of as an example of Noether's theorem.

3.2.2 Solving the geodesic equation

We now want to solve the geodesic equation. I will sometimes write the 4-velocity as $u^\mu = \dot{x}^\mu$, where an overdot denotes a derivative with respect to s. Now recall from the first half of the course that if we are solving the geodesic equation, we always pick the parameter s so that the velocity along the worldline has magnitude 1, i.e.

$$g_{\mu\nu}u^\mu u^\nu = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -\epsilon \qquad \epsilon = 1, 0 \tag{3.34}$$

and where ϵ is a *constant* that we take to be either 1 for timelike paths (i.e. that paths followed by massive particles) or 0 for null rays (i.e. the path followed by light rays). This remind you, this can always be arranged by a choice of s, and a choice of s for which this is true is called an *affine parametrization.*.

With this choice of s, the geodesic equation becomes:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{3.35}$$

We can now explicitly calculate the θ equation to be

$$2\frac{d}{ds}\left(r^2\dot{\theta}\right) = 2r^2 \sin\theta\cos\theta\dot{\phi}^2 \tag{3.36}$$

(we can get this e.g. from the Christoffels). Note now that $\theta(s) = \frac{\pi}{2}$, $\dot{\theta}(s) = 0$ is a solution to this equation. Physically, this means that if the orbit starts out in the equatorial plane, it will *stay* in the equatorial plane. This is a consequence of the conservation of the *direction* of angular momentum.

Now we *could* write out the other Euler-Lagrange equations for (t, r, ϕ); however this is a pain. We will instead use conserved quantities. In fact, there is almost always a 2-step process to solving geodesic equations that we use:

1. **Identify the conserved quantities**. In our case there are two obvious Killing vectors. One is time translation:

   $$H^\mu = (\partial_t)^\mu = (1, 0, 0, 0) \tag{3.37}$$

   which leads to the conserved quantity

   $$K = H_\mu u^\mu = \left(1 - \frac{2GM}{r}\right)\dot{t} \tag{3.38}$$

   You can think of this is as the energy of the particle. Is it kinetic energy or potential energy? In general relativity, this distinction does not really make sense – but nevertheless it is a conserved quantity.

   The other Killing vector is associated with translations of ϕ, or rotations around the 2-sphere:

   $$R^\mu = (\partial_\phi)^\mu = (0, 0, 0, 1), \tag{3.39}$$

   whose associated conserved quantity is angular momentum:

   $$J = R_\mu u^\mu = r^2 \sin^2\theta\dot{\phi} = r^2\dot{\phi} \tag{3.40}$$

   where the last equality holds because we are restricting to the equatorial plane $\theta = \frac{\pi}{2}$. Note that if $J \neq 0$ then the particle is always moving in ϕ.

2. **Use the normalization of the four-velocity.** Remember that we have

   $$\left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\dot{r}^2 - r^2(\dot{\phi}^2) = \epsilon \tag{3.41}$$

Note now that we can use the conservation laws to express $\dot{t}$ in terms of K and $\dot{\phi}$ in terms of J. This leads to

$$\frac{K^2}{\left(1-\frac{2GM}{r}\right)} - \frac{\dot{r}^2}{\left(1-\frac{2GM}{r}\right)} - \frac{J^2}{r^2} = \epsilon \tag{3.42}$$

This is already nice; instead of having three functions of s, we now only have one: $r(s)$. It is easier to understand what this equation means if we rearrange it into the following form

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2GM}{r}\right)\left(\frac{J^2}{r^2}+\epsilon\right) = \frac{1}{2}K^2 \tag{3.43}$$

This is useful because it is of exactly the form for the conservation of energy of a particle moving in a 1d potential $V(r)$ with an effective energy E:

$$\boxed{\frac{1}{2}\dot{r}^2 + V(r) = E \qquad V(r) = \frac{1}{2}\left(1-\frac{2GM}{r}\right)\left(\frac{J^2}{r^2}+\epsilon\right) \qquad E = \frac{1}{2}K^2} \tag{3.44}$$

Note that E here is just a name for the effective energy in this potential. As we have a lot of intuition for such things, we can now plot the potential $V(r)$ and understand what is happening with the particle.

For the rest of this section we will set $\epsilon = 1$ and study timelike geodesics.

Note that if we expand it out we get

$$V(r) = \frac{1}{2}\left(\epsilon - \frac{2GM\epsilon}{r} + \frac{J^2}{r^2} - \frac{2GMJ^2}{r^3}\right) \tag{3.45}$$

It is an interesting fact that if we were studying the orbits of planets in ordinary Newtonian gravity, we would have gotten exactly the same equation (3.44) except that the $\frac{J^2}{r^3}$ term at the end would have been missing. Note that this term is not very important far away from the center (and hence again we recover Newtonian physics at weak-fields, etc. etc.).

A plot of the potential for a typical value of J is shown in Figure 3.1. At $r = 2GM$ the potential is always zero. This is a signature of the black hole; we will come back to this. If we had been doing Newtonian gravity instead the potential would have gone up forever as we approached $r = 0$. The precise shape depends on the value of the angular momentum.

The behavior of the particle depends on the value of the energy K. The different sorts of orbits are shown in Figure 3.2. For example, if K is such that the particle sits exactly at the bottom, then we have a circular orbit with radius r_c (case A).

If we now increase the energy a little bit, then r can vary, and we have a non-circular orbit. I want to point out that it actually *isn't* a pure ellipse! (case B). We will come back to this later.

If we increase the energy still further, then its a bit different – it is no longer a *bound* orbit, instead the particle comes in from infinity, comes up to a minimum radius r_{min}, and then goes out again; note there is no maximum r any more (case C)

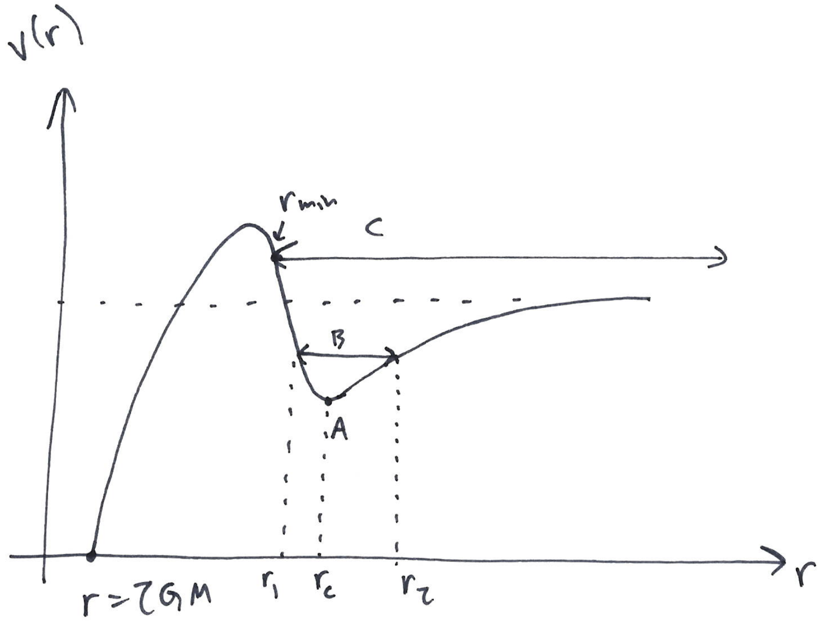

Figure 3.1: Effective potential $V(r)$

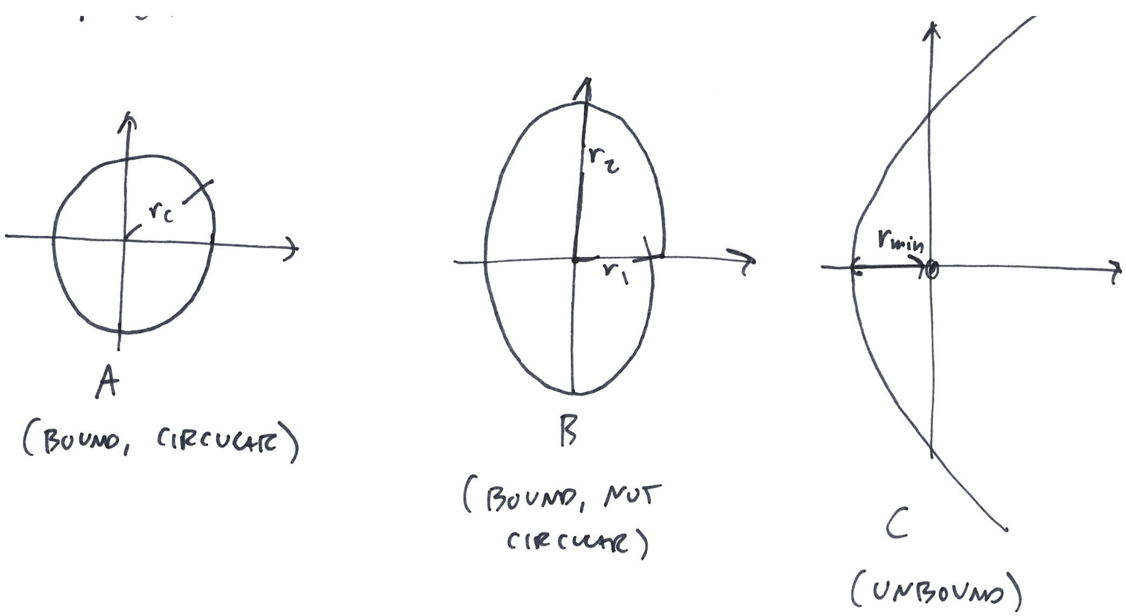

Figure 3.2: Different sorts of possible orbits around Schwarzschild metric.

3.2.3 Circular orbits

Let's be a bit more quantitative. When can we have a circular orbit? Only when $V'(r) = 0$. We directly calculate

$$\frac{dV}{dr} = \frac{1}{r^4}\left(-J^2 r + GM(3J^2 + r^2)\right) \tag{3.46}$$

Thus we have a stationary point r_c when we have a solution to the quadratic equation above, which turns out to be at

$$r_c = \frac{J^2 \pm \sqrt{J^4 - 12G^2M^2J^2}}{2GM} \tag{3.47}$$

So when J is big enough, there are two roots to this equation; one of them is a stable circular orbit and the other is an unstable one. **See pictures.** If we make J very large then we can see that

$$r_c(J \to \infty) \approx \left(\frac{J^2}{GM}, 3GM\right) \tag{3.48}$$

The stable one is further out; it is analogous to the stable orbits that you have studied before in Newtonian physics (and indeed its radius is given by the usual Newtonian formula). The unstable one is a new thing from general relativity. It corresponds to something orbiting around, but the slightest touch on it will either send it spiralling into the black hole or out to infinity.

Now let's start decreasing J. If we make J smaller and smaller, the two roots get closer and closer together. Eventually the argument of the square root is zero, and the two roots collide and vanish. This happens at

$$J = 2\sqrt{3}GM \tag{3.49}$$

for which we have

$$r_c = 6GM \tag{3.50}$$

This is thus the location of the innermost stable circular orbit, or ISCO. You cannot orbit stably any closer than this, although you can orbit unstably up to $r = 3GM$.

Note that it is thus totally clear that for sufficiently small angular momentum you will just fall straight into the black hole, because the potential is monotonic. This makes sense; essentially for small J the angular momentum is small.

One can repeat this analysis for light rays; we'll do this in the next section.

3.3 Solar system tests

Now we will discuss how the physics of the Schwarzschild metric can result in actual *observable* consequences. We will discuss three experimental observations; the deflection of light by the gravitational field, the red-shift of light, and the precession of the perihelion of Mercury.

3.3.1 Deflection of light

We know that light falls towards gravitational fields.

Here we will calculate precisely how much it falls. From the picture we see that we care about the dependence of ϕ on r. For null geodesics from before (setting $\epsilon = 0$) we have

$$J = r^2\dot{\phi} \qquad \dot{r}^2 = K^2 - \frac{J^2}{r^2}\left(1 - \frac{2GM}{r}\right) \tag{3.51}$$

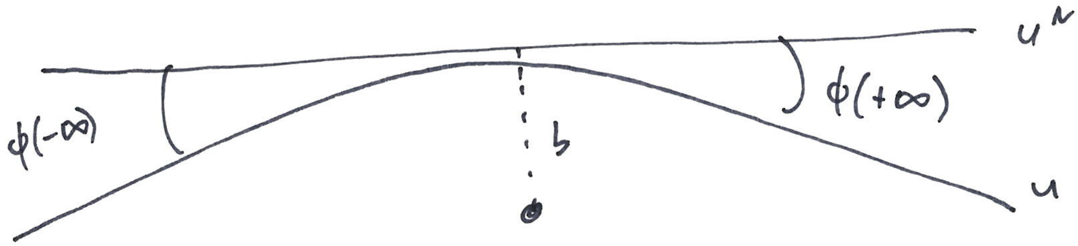

Figure 3.3: Deflection of light by a gravitational field

Now define the following quantities:

$$u \equiv \frac{1}{r} \qquad b \equiv \frac{J}{K} \tag{3.52}$$

b is called the "impact parameter"; we will see why in a second. Now we can work out

$$\frac{du}{d\phi} = \frac{du}{ds}\frac{ds}{d\phi} = -\frac{\dot{r}}{r^2\dot{\phi}} = \frac{\dot{r}}{J} \tag{3.53}$$

Plugging in the value of $\dot{r}$ from above and manipulating a little bit, we find

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{b^2} - u^2 + 2GMu^3 \tag{3.54}$$

As always, any real-life equations are too hard to solve. We must develop some kind of expansion. Note that the last term here is the contribution from general relativity; we may assume that it is small.

For a second, suppose we drop it. Then the Newtonian equation is

$$\left(\frac{du^N}{d\phi}\right)^2 = \frac{1}{b^2} - (u^N)^2 \tag{3.55}$$

whose solution is

$$u^N = \frac{1}{b}\sin\phi\ . \tag{3.56}$$

Here N stands for "Newtonian". This is a straight line in polar coordinates, whose closest approach to the sun is b (the "impact parameter"), as shown in Figure 3.3. Note that it seems that Newton would say that light doesn't fall at all. Now we see what Einstein has to say: we will look for a solution of the form

$$u = \frac{1}{b}\sin\phi + v \tag{3.57}$$

where v is small. Here "small" means that we assume that

$$v \sim \mathcal{O}(G), \tag{3.58}$$

just like the GR term above. Plugging this into (3.53) we find

$$\left(\frac{1}{b}\cos\phi + v'\right)^2 = \frac{1}{b^2} - \left(\frac{1}{b}\sin\phi + v\right)^2 + 2GM\left(\frac{1}{b}\sin\phi + v\right)^3 \tag{3.59}$$

Now we keep terms that are only $\mathcal{O}(G^1)$; this means we can throw away terms like v^2, vG, v^3, etc. etc. Note that all terms that are $\mathcal{O}(G^0)$ cancel because we picked u^N to satisfy the leading equation! After all the dust settles we find the following first order ODE:

$$(\cos\phi)\, v' + (\sin\phi)\, v = \frac{GM}{b^2}\sin^3\phi \tag{3.60}$$

We can solve this with an integrating factor. I know that the answer is

$$v(\phi) = \frac{GM}{b^2}\left(1+\cos^2\phi\right) \tag{3.61}$$

(Check if you like!)

Now we need to figure out what this means; refer to Figure 3.3. We need to determine the angle $\phi(\infty)$; at $r\to\infty$, which means that $u\to 0$. We can then expand the cos and sin using their Taylor expansions around $\phi = 0$:

$$\sin(\phi\to 0) \approx \phi + \cdots \qquad \cos(\phi\to 0)\approx 1-\frac{1}{2}\phi^2+\cdots \tag{3.62}$$

So from the geometry we see that at infinity

$$u = 0 = \frac{1}{b}\sin\phi + \frac{GM}{b^2}\left(1+\cos^2\phi\right) \approx \frac{\phi}{b} + \frac{2GM}{b^2} \tag{3.63}$$

which means that

$$\phi(\infty) = -\frac{2GM}{b} \tag{3.64}$$

We get an equal contribution from the other end; thus the full angular deflection is

$$\boxed{\Delta\phi = \frac{4GM}{b}} \tag{3.65}$$

Now suppose we want to measure this! What do we do? The biggest mass around is the sun, we should use that; we can wait for stars to pass behind it and see if we can measure the deflection of the light. The signal is biggest when the light passes right by the sun (so that the impact parameter is the radius of the sun itself.) At that point we find

$$\Delta\phi_{sun} = 8.45\times 10^{-6}\ \text{rad} \tag{3.66}$$

or 1.75 arcseconds, as they say. This is small, but visible. There is only one problem; you can't see the stars when they are behind the sun.

So you wait for an eclipse! And on May 29, 1919 that is precisely what Sir Arthur Eddington did, from the island of Principe off the coast of West Africa. And he took pictures of the sky. And lo and behold, if you compare where each star *is* with where it *normally* is, indeed they appear "askew" by precisely this amount.

It's really quite amazing.

3.3.2 Precession of the perihelion of Mercury

As we have discussed earlier, orbits in Newtonian gravity are a bit special; due to the fact that the attractive Newtonian potential is exactly $\Phi \sim \frac{1}{r}$, orbits close exactly, and bound orbits form ellipses. This is not true any longer in general relativity; the attractive force differs slightly from that of Newtonian gravity, and thus we expect the perilhelion of orbits to *precess*, as shown in Figure 3.4. The calculation isn't too bad but is a little boring to do on the board, so in the homework for next week you will calculate this and show that it is

$$\delta\phi \approx \frac{6\pi(GM)^2}{J^2} \tag{3.67}$$

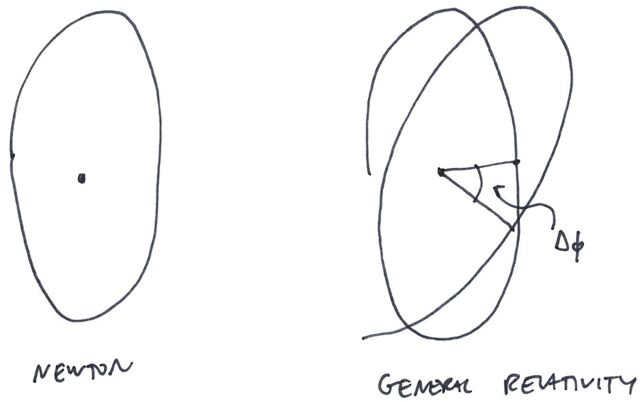

Figure 3.4: In Newtonian physics, orbits are closed ellipses. In general relativity, the perihelion precesses (for solar system applications, the picture is greatly exaggerated).

Now let's turn to real life. Astronomers have kept very close track of the movement of planets, and it was a fact in the early twentieth century that the perihelion of Mercury was observed to advance 5600 arc seconds per *century*. Now this can actually come from a number of different sources; for example, all the other planets pull on it (Jupiter contributes 153" or so). But after you subtract off all of the known effects, there are 43" left over.

This had happened before; there was a similar problem with Uranus, and from this a 17th century astronomer *predicted* the existence of Neptune. Similarly, people thought that there was another planet called *Vulcan* between the sun and Mercury. But oddly, no one ever saw it.

So now, go home and plug in the numbers into the formula above; you find exactly 43". Too bad for Vulcan, but hooray for GR.

3.3.3 Redshift of light

Imagine that you are a ray of light, or a particle of light (called a photon) trying to escape from a gravitational field. You are clawing your way up; you will lose energy because of the attraction of gravity. As you're a photon this does not slow you down; but because of quantum mechanics we know that $E = \hbar\omega$, and thus your frequency will go down. This is called the *redshift* of light.

Let me say the same thing in different language. Let's imagine that you are at a radius r_i and you are Skyping with someone at r_o; your friend will see you moving in *slow motion*, by the arguments before. This is gravitational time-dilation. Let's derive this effect.

Consider two observers, each at fixed radius r_i and r_o and the same θ, ϕ. Their proper times are s_i, s_o. Note that the worldline of the inner observer has

$$\left(1 - \frac{2GM}{r_i}\right)\left(\frac{dt_i}{ds}\right)^2 = 1 \qquad \rightarrow \qquad t_i = \left(1 - \frac{2GM}{r_i}\right)^{-\frac{1}{2}} s_i \tag{3.68}$$

We have exactly the same equation for the outer observer with $i \to o$.

Now suppose the inner observer sends a light ray. We will imagine that he sends two light rays with a difference in proper time Δs_i, and we will see what the observed difference in proper times Δs_o is at the

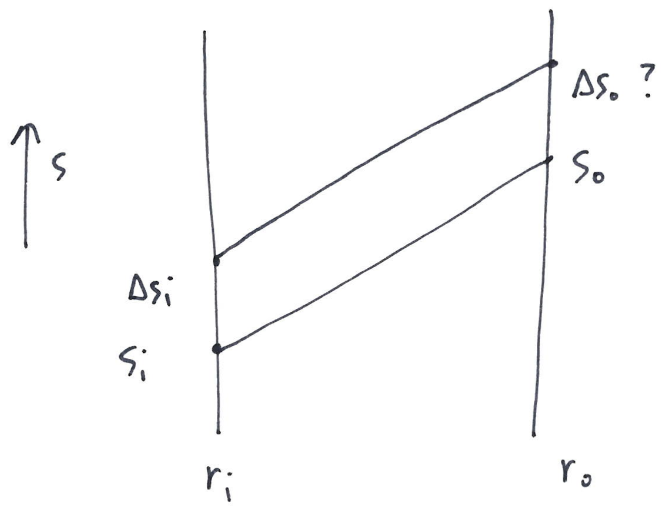

Figure 3.5: Geometry for deriving the redshift of light.

other end. The setup is shown in Figure 3.5.

The light rays travel along a null geodesic

$$\left(1-\frac{2GM}{r}\right)dt^2=\left(1-\frac{2GM}{r}\right)^{-1}dr^2 \qquad \to \qquad \frac{dt}{dr}=\frac{1}{\left(1-\frac{2GM}{r}\right)} \tag{3.69}$$

which we can integrate to find that

$$t=f(r)=\int^r dr'\frac{1}{\left(1-\frac{2GM}{r'}\right)} \tag{3.70}$$

It turns out we won't actually need the form of $f(r)$. So the point is that along the geodesic $t-f(r)$ is constant along the geodesic. So if the signal is sent at t_i and received on the other end at t_o, then we have

$$t_i=t_o-f(r_o)+f(r_i) \tag{3.71}$$

Now using (3.68) we find that the difference in *proper* times is

$$\left(1-\frac{2GM}{r_i}\right)^{-\frac{1}{2}}s_i=\left(1-\frac{2GM}{r_o}\right)^{-\frac{1}{2}}s_o-f(r_o)+f(r_i) \tag{3.72}$$

From here we can immediately read off what Δs_o is from Δs_i. I'll write the answer in terms of the red shift of frequencies, which is the inverse of the shift in proper times:

$$\boxed{\frac{\omega_o}{\omega_i}=\frac{\Delta s_i}{\Delta s_o}=\sqrt{\frac{1-\frac{2GM}{r_i}}{1-\frac{2GM}{r_o}}}} \tag{3.73}$$

Ok; so note that if $r_o>r_i$, then the observed frequency is *less.*. In other words, near the earth, if you hold your cell phone below you so that the photons have to climb up to reach your face, the screen should look *redder*. Can you see it? Well, no. The effect is tiny. Nevertheless, it was done in an experiment in

Cambridge, Massachusetts by Pound and Rebka. The height of the tower that they used (the physics building at Harvard) was $\Delta r = 22$m, so the signal they were looking for was a redshift of

$$\frac{\omega_o}{\omega_i} \approx 1 - \frac{GM}{r^2}\Delta r \approx 1 - 2.4 \times 10^{-15} \tag{3.74}$$

The fractional redshift is *tiny*.

3.4 The Black Hole Horizon

In this section we now finally approach the Schwarzschild radius at $r = r_s \equiv 2GM$. Note that we can already see that something dramatic is happening; let us imagine that we have two observers (and let us call them Romeo and Juliet), and Romeo sits at r_o above while Juliet is at r_i. As Juliet approaches $r = 2GM$, we see that Romeo perceives her moving more and more slowly, i.e. the red-shifted frequency is going to *zero*. So it is clear that something dramatic is happening.

Now we try to understand what happens. Note that the metric is

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{3.75}$$

There are two points where something bad seems to happen: at $r = 2GM$ (called the *event horizon*) and at $r = 0$. These are two different kinds of singularity:

Coordinate singularity: The geometry is well-behaved at that point, but that we chose coordinates badly. Coordinate singularities can be removed through a redefinition of coordinates. We will repair the singularity at $r = 2GM$.

Curvature singularity: a place where the geometry is actually singular, e.g. infinitely curved. No redefinition of coordinates can cure it. One way to exhibit a curvature singularity is by finding a scalar curvature invariant that blows up. We can't use the Ricci scalar R (because it is zero), but we could use e.g.

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6} \tag{3.76}$$

which is a scalar that blows up at $r = 0$ (but, notice, not at $r = 2GM$). Thus $r = 0$ is actually bad; as this is a scalar, it is the same in all coordinates, and no redefinition will fix it.

3.4.1 Eddington-Finkelstein coordinates

We move on to the coordinate singularity. To understand what is happening we need to revisit the idea of *light cones*. Consider for example flat space:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{3.77}$$

Think about light rays coming from $r = 0$; they all follow $t = \pm r$, which looks like Figure 3.6.

Note that timelike trajectories – that is, people, rocket-ships, etc. – must stay in the interior of the light cone. Thus the light cones define the boundary between where you *can* and *cannot* go.

We will now try to build light cones for the Schwarzschild metric. This is found from solving

$$-\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 = 0 \qquad \to \qquad \frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1} \tag{3.78}$$

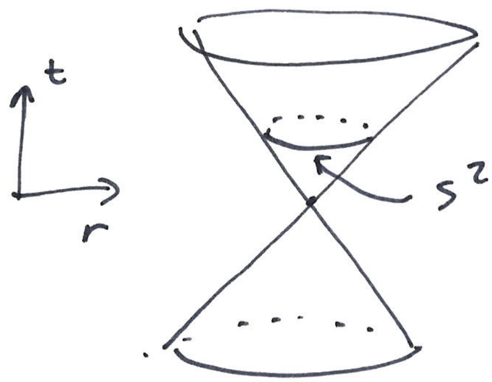

Figure 3.6: Light cones in flat space; they follow $t = \pm r$, and each circle shown is actually an S^2

Note that as we approach the horizon, the $\frac{dt}{dr}$ is diverging, and thus light cones are becoming more and more narrow! Something odd is happening. Note that as Juliet heads in, her signals find it harder and harder to reach Romeo.

However her light rays also find it harder and harder to move in; it turns out that this is an artifact that has to do with a lousy choice of the r coordinate. A way to repair this is to "follow the geodesics". The first step is to integrate (3.78) to find

$$t = \pm r^* + \text{const} \qquad \frac{dr^*}{dr} = \left(1 - \frac{2GM}{r}\right)^{-1} \tag{3.79}$$

which we can easily integrate to find

$$r^* = \int^r dr' \left(1 - \frac{2GM}{r'}\right)^{-1} = r + 2GM \log\left(\frac{r}{2GM} - 1\right) \tag{3.80}$$

r^* is just a different radial coordinate that is naturally adapted to the black hole horizon. Note that the horizon is at $r^* = -\infty$ – that's why you sometimes hear r^* called the "tortoise coordinate."

Now we adopt new coordinates that are well suited to these light rays.

$$v = t + r^* \tag{3.81}$$
$$u = t - r^* \tag{3.82}$$

An *ingoing* light ray satisfies $v = \text{const}$ and an outgoing light ray satisfies $u = \text{const}$. Now we will use v and r as coordinates. We need to rewrite the metric; using:

$$dt = dv - \frac{dr^*}{dr} dr = dv - \left(1 - \frac{2GM}{r}\right)^{-1} dr \tag{3.83}$$

We find that the Schwarzschild metric becomes

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2} \tag{3.84}$$

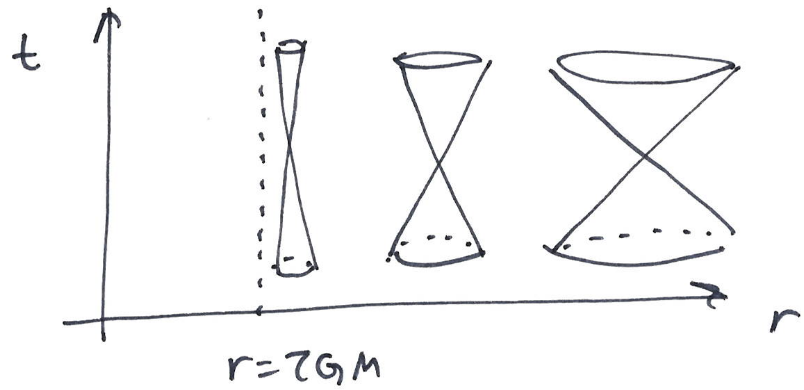

Figure 3.7: Light cones in Schwarzschild coordinates (t, r)

These are called *ingoing Eddington-Finkelstein coordinates.* Here actually nothing bad happens at all at $r = 2GM$; the dv^2 term vanishes, but this does not matter. Note that light rays satisfy

$$\left(1 - \frac{2GM}{r}\right) dv^2 - 2dvdr = 0 \tag{3.85}$$

which means

$$\frac{dv}{dr} = 0 \qquad \frac{dv}{dr} = 2\left(1 - \frac{2GM}{r}\right)^{-1} \tag{3.86}$$

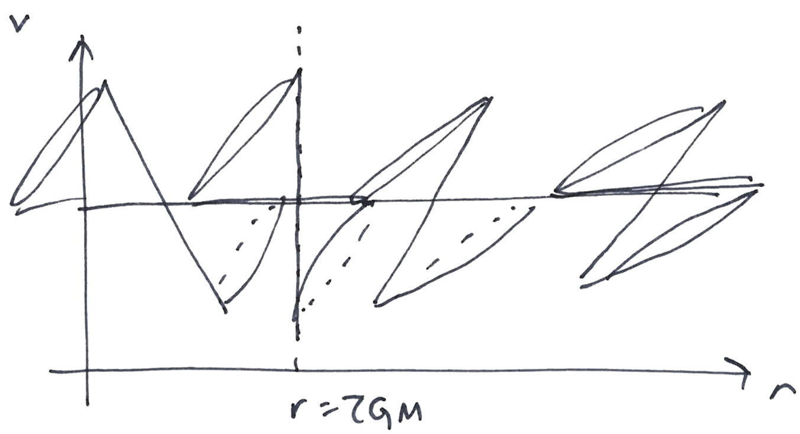

Figure 3.8: Light cones in Eddington-Finkelstein coordinates (c, r)

Thus we see that the light cones are *tipping*, as shown in Figure 3.8! Now we can see precisely what is happening. For $r > 2GM$ the light cones permit null rays to go both towards positive r and towards negative r; light can both go in and come out. But as we approach $r = 2GM$, it begins to *tip*, and at $r = 2GM$, one of the edges becomes *vertical*; this means that light can just barely make it out.

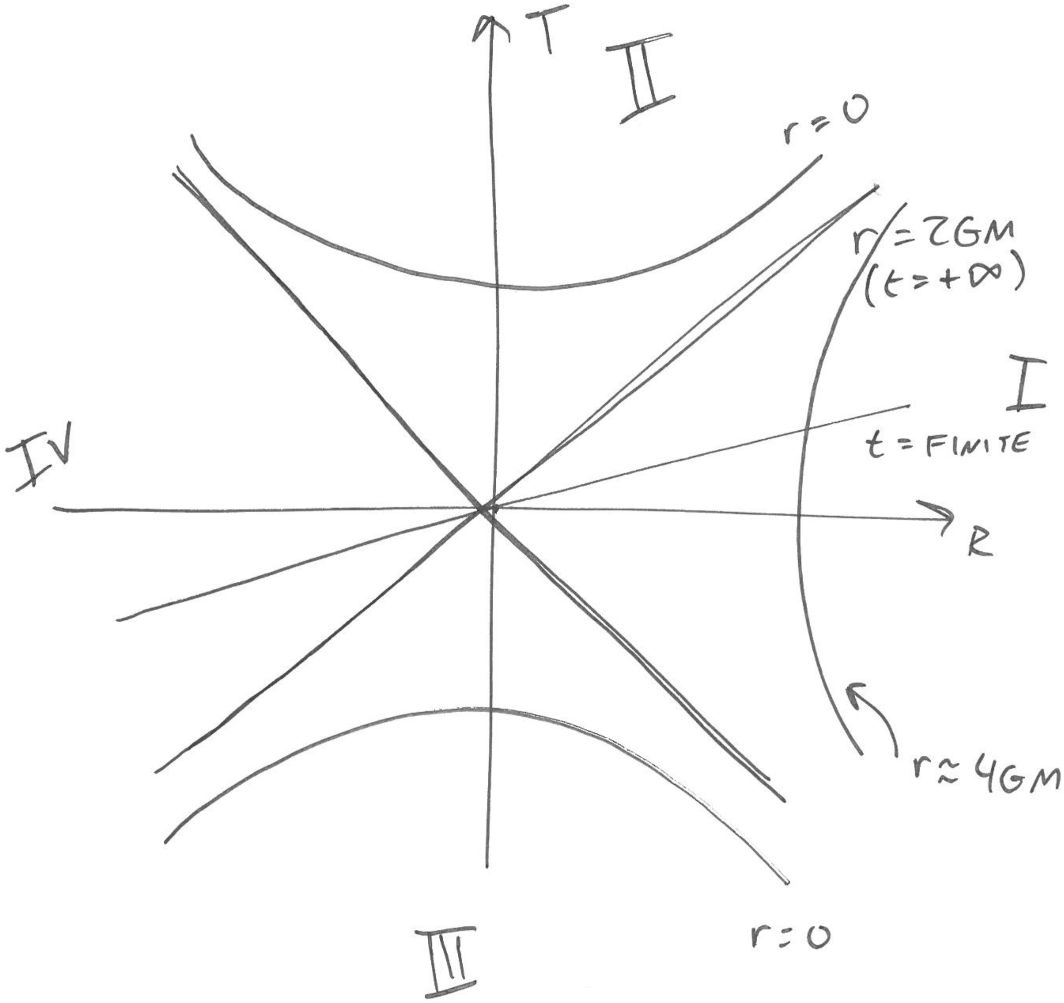

Figure 3.9: Maximally extended black hole in Kruskal-Szerkes coordinates.

For $r < 2GM$, all is lost; the light cones point towards negative r. This means that light can no longer come out of the black hole; it only falls in. Of course, all timelike trajectories are bounded by the light cone: thus they will also all move towards the interior.

So once Juliet crosses the event horizon at $r = 2GM$, all is lost: the very causal structure of spacetime will not permit her to escape, and no matter the strength of her rocket engines, she can no more make it out to Romeo than she can go back in time. The event horizon may be defined as the surface across which nothing can escape back out to infinity.

Finally, note that the fact that all timelike trajectories move towards decreasing r basically means that r has become a timelike coordinate; we could have guessed this from the original form of the metric, as its clear that the sign of the dr^2 term and the dt^2 term switch when we cross the horizon.

3.4.2 Kruskal-Szerkes coordinates

Recall that we followed ingoing geodesics into the black hole, and we found that we could change coordinates to get a smooth metric across the horizon. As it turns out, we get a similar answer if we follow *outgoing* geodesics; though it is not obvious, it turns out you end up in a *different place*. To explain what this means, and to help explain the causal structure of the spacetime behind the horizon and the horizon itself, we will do yet another change of coordinates, to so-called Kruskal-Szerkes coordinates (T, R).

The transformation is the following:

$$T = \left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} \exp\left(\frac{r}{4GM}\right) \sinh\left(\frac{t}{4GM}\right) \tag{3.87}$$

$$R = \left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} \exp\left(\frac{r}{4GM}\right) \cosh\left(\frac{t}{4GM}\right) . \tag{3.88}$$

In terms of these coordinates the metric becomes (after some algebra, which you can check at home)

$$\boxed{ds^2 = \frac{32G^3M^3}{r} e^{-\frac{r}{2GM}} \left(-dT^2 + dR^2\right) + r^2 d\Omega^2} \tag{3.89}$$

Note that there are still some r's floating around; we should imagine that r is determined implicitly from T, R by

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{\frac{r}{2GM}} \tag{3.90}$$

These coordinates cover the whole spacetime nicely and so illuminate many of its confusing features. Importantly, note that light rays in these coordinates follow the lines

$$T = \pm R \tag{3.91}$$

So they follow straight lines; this makes the causal structure easy to understand.

Note also that the horizon at $r = 2GM$ looks perfectly normal and in fact boring; nothing in particular is happening to the metric (3.89) there. This means that we are completely free to follow the (T, R) coordinates past the horizon.

So let's draw a picture of the spacetime in T, R coordinates. First, let's note that

$$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right) \tag{3.92}$$

So a constant t-line maps to a straight line in the T, R plane: interestingly, as t runs from $-\infty$ to $+\infty$, the slope runs from -1 to 1.

How about constant-r? From (3.90) we see that constant-r slices map to hyperbolae in the T, R plane. As we approach $r = 2GM$, these hyperbolae degenerate and become the two straight lines:

$$T^2 - R^2 = 0 \qquad r = 2GM \tag{3.93}$$

Note also that the *singularity* at $r = 0$ is

$$T^2 - R^2 = 1 \qquad r = 0 \tag{3.94}$$

Putting all of this together, we end up with the following picture for what is called the Eternal Black Hole, or the Maximally Extended Schwarzschild Spacetime. There is a lot going on here, so let's discuss it carefully. We call the regions in the picture I, II, III, and IV. So let's discuss them all in turn:

1. **Region I**: this is $r > 2GM$, $-\infty < t < \infty$, so *outside the horizon*. This is the region that is covered properly by the original coordinate system. Let's first look at the worldline of someone hovering outside the horizon at fixed r. **Picture**. Note that as the original time coordinate t goes from $-\infty$ to $+\infty$, we stay in this quadrant. This is what we mean when we say that the coordinate system only covers the region outside the horizon.

2. **Horizon**: now the horizon is $r = 2GM$, which is these two lines. Let us focus on the part adjoining Region I for now. Note that it comes in two halves, called the past horizon and the future horizon. Note also that the horizon is *not* a timelike surface, unlike any other constant r surface outside the horizon; instead it is *null*, i.e. of its three tangent vectors, two are spacelike (the θ, ϕ directions), and the other is null. Now if we take a timelike trajectory and follow it into the future through the horizon, eventually we end up in:

3. **Region II**: This is where you end up if you fall into the black hole, or if you follow an ingoing null geodesic like we did last lecture. This is where Juliet ended up last time. Note that in this picture it is especially clear that you can't come out of the horizon again, because the light cones are very easy to understand. Next, note that once you're inside, any timelike trajectory will ultimately hit the singularity at $r = 0$.

   Let us dwell on this for a moment. Not only can you not escape to the outside of region I, you can't even help yourself – you will always be drawn towards the singularity. This is because the singularity is *not* a point in space – it is a spacelike surface, and it is thus a *moment in time*, which should be clear from the picture. Once you enter the horizon, you can no more escape the singularity than you can cease to grow older. That is why the singularity is such a horrifying thing.

   It is possible to calculate how much proper time you have once you enter – it depends on how quickly you fall in, but the answer is always of the order of GM.

4. **Region III**: This is the *time-reverse* of Region II. It is there because our metric is symmetric under time-reversal; thus whatever is there in the future is also there in the past. You may think of it as a region from which stuff always comes out and never goes in. This is where you end up if you follow outgoing null geodesics through the horizon rather than ingoing ones like we did.

5. **Region IV**: This might be the most confusing of all. What is this? Naively it looks like something that is precisely like Region I; but remember, region I was the entire region of asymptotically flat space outside the event horizon. This is the same: it is an *entirely different region of asymptotically flat space* that is connected to this one. The Schwarzschild metric does not actually describe a hole in spacetime; it represents it two different spacetimes that are connected. The connection between the two regions is called the Einstein-Rosen bridge. If I were to draw a picture of a spatial slice at this time, it looks like Figure 3.10.

   Can you move from Region IV to Region I? If we could, it would be great! We could use these bridges to travel faster than light, Star Trek, Interstellar, etc. etc. Actually however you cannot; this is clear from the picture, once you enter this region you always hit the singularity. The Einstein-Rosen bridge is a lousy bridge, in that geodesics connecting the two sides are always spacelike. Too bad.

   There are theorems that prove that you cannot have a traversable wormhole given some reasonable conditions on the stress energy tensor $T_{\mu\nu}$ (basically that energy is always positive).

This is the full structure of the Schwarzschild metric. Note that from here we can now finally draw a global picture of the fate of Romeo and Juliet. If Juliet falls into the black hole, her worldline goes like this, and Romeo's looks like this. **Picture**.

Now let me describe in words what Romeo *sees*; he can never actually see Juliet fall in, because any light ray she sends as she crosses never makes it out! This means that her last few signals will be stretched out, taking longer and longer to reach him. From his point of view, he sees her moving more and more slowly,

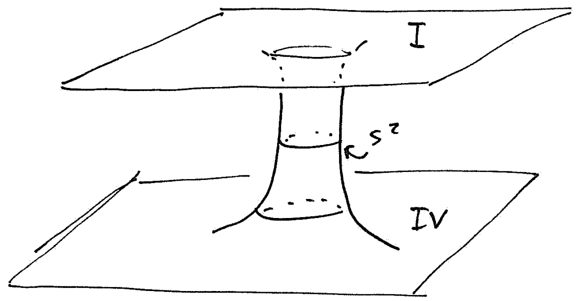

Figure 3.10: Picture of spatial topology at $T = 0$ for maximally extended spacetime

until she freezes entirely on the black hole horizon. For all of eternity, as long as he lives, she will remain frozen on the horizon. Romantic.

From her point of view, on the other hand, she simply falls in. It takes her finite proper time. She hits the singularity and dies. Let us now calculate how exactly she dies. Note that we assume that she follows a geodesic, but it turns out that the gravitational force on her feet will be much stronger than that on her head, eventually pulling her apart.

Recall from the first term the *equation of geodesic deviation*, which says that if you have two neighbouring geodesics separated by a vector S^μ where U^μ is the velocity along the geodesics, then

$$U^\mu \nabla \mu \left(U^\lambda \nabla_\lambda S^\nu \right) = R^\nu{}_{\rho\mu\lambda} U^\mu U^\rho S^\lambda \tag{3.95}$$

Take now S^μ to be a vector pointing from the head of Juliet to her feet, as in Figure 3.11. The way to understand this is that

$$\frac{d^2}{ds^2} S \sim R \times S \tag{3.96}$$

i.e. the acceleration of S is roughly measured by the Riemann tensor, which tells you the acceleration gradient that Juliet feels.

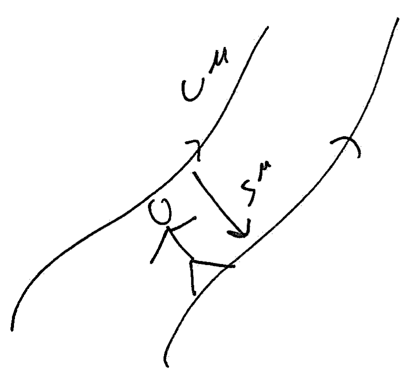

Figure 3.11: Juliet falls into a black hole while a vector S^μ points from her head to her feet.

In other words, we need to ask how much of an acceleration gradient can a human being handle before being pulled apart? If you google this, it says that 65 g's of acceleration will kill someone. Over Juliet's 2 m person, this corresponds to an acceleration *gradient* of

$$\frac{65 \times 9.8\text{m s}^{-2}}{2\text{m}} \approx 300\text{s}^{-2} \tag{3.97}$$

We should set this equal to the Riemann tensor, whose entries typically have magnitude

$$R_{...} \sim \sqrt{48}\left(\frac{GM}{r^3}\right) \tag{3.98}$$

Putting in the numbers for a solar mass black hole, we find that the force is fatal when

$$r_{\text{sun, fatal}} \approx 1400\text{km} \tag{3.99}$$

But r_s for a solar mass black hole is 3km; thus Juliet does not even make it to the horizon before she gets ripped apart.

What if we took a bigger black hole? There is a black hole at the center of the galaxy with mass four million times that of the sun. It is called Saggitarius $A*$. If we use that we find,

$$r_{A^*,\text{fatal}} \approx 240,000\text{km} \tag{3.100}$$

which should be compared to its Schwarzschild radius, which is about 12 million km. So there you can make it pretty far inside before you are killed by the tidal forces.

3.4.3 Other kinds of black holes

The picture above is amazing. But does it really exist? Do black holes exist in real life?

There are thought to be many. Essentially a star in the sky is a controlled explosion; it is a nuclear bomb that is trying to explode while gravity keeps it in. Once it runs out of fuel, then gravity takes over and pulls it inwards. If the star is big enough, it can collapse to become a black hole. One of these is at centre of our galaxy, called Saggitarius A*.

Do those look like the picture? Actually, they do not. The reason is that in the past the spacetime is replaced by the star region, so there is no past horizon. There is also no other spacetime connected to it. Thus we only have one half and the future horizon.

There is a lot more that can be said about black holes. I will just mention some more quick facts without extensive derivation. It is possible to have a *charged* black hole; this is a solution to Einstein's equations coupled to electromagnetism. You will recall the stress tensor for electromagnetism from Lecture 4 of this term. Putting it in the Einstein equation we find:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{4\pi}\left(-g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\rho}\right) \tag{3.101}$$

It is possible to solve this; I will not do this here, but the answer is called the Reissner-Nordstrom black hole, and its metric is:

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r} + \frac{G}{4\pi}\frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{G}{4\pi}\frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad F_{rt} = \frac{Q}{r^2}} \tag{3.102}$$

Here Q is the charge of the black hole. Note that the electric field carries energy, and that energy sources spacetime curvature; that is why the metric depends on Q.

There are also *Kerr black holes*, which are black holes that are rotating and so carry angular momentum J. They have a metric that is a little intricate, so I wrote it down already.

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{4GMar\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)^2 + a^2\Delta\sin^2\theta\right]d\phi^2 \tag{3.103}$$

where the different things are

$$\Delta(r) = r^2 - 2GMr + a^2 \qquad \rho^2(r,\theta) = r^2 + a^2\cos^2\theta \qquad a = \frac{J}{M} \tag{3.104}$$

J is the angular momentum. You can also have both charge *and* angular momentum.

Now you might ask: is there anything else? Are there other kinds of black holes in our universe? As a matter of fact, there are *not*. There is something called the **No-Hair Theorem**, which says that

Time-independent, asymptotically flat black hole solutions to general relativity coupled to electromagnetism are fully characterized by the parameters of mass, electric charge, and angular momentum.

In other words: a black hole depends only on three numbers. This should be compared with the situation for ordinary objects: the spacetime outside the earth, for example, depends on all the details of the surface of the earth, its mountains and valleys and mass distributions, etc. Relativists call this sort of data the "hair" of the solution. This is simply *not* the case for a black hole; it does not depend on anything but its mass, charge, and angular momentum.

This is a very baffling statement. It suggests that black holes are in many ways the "simplest" objects in physics, because what I have written down is the *full story*; there is nothing else. I will close the chapter on black holes by quoting how the eminent astrophysicist S. Chandrasekhar felt when he realized this:

In my entire scientific life, extending over forty-five years, the most shattering experience has been the realization that an exact solution of Einstein's equations of general relativity, discovered by the New Zealand mathematician, Roy Kerr, provides the absolutely exact representation of untold numbers of massive black holes that populate the universe. This shuddering before the beautiful, this incredible fact that a discovery motivated by a search after the beautiful in mathematics should find its exact replica in Nature, persuades me to say that beauty is that to which the human mind responds at its deepest and most profound.

4 Cosmology

We now turn to a study of the universe at the longest possible scales – i.e. *cosmology*, or a study of the universe as a whole, e.g. we know that it is expanding, and this is what we will now describe, calculating things like how the universe began, etc. etc.

First, a few words on units and scales. The basic astronomer unit is the *parsec*, with 1 parsec $= 3 \times 10^{16}$ m or 3.26 light years. To give you a sense of scale, the distance between galaxies is on the order of megaparsecs (Mpc), and the observable region of the universe is about 10^4 Mpc. In cosmology we study physics on scales that are of a typical size 100 Mpc or more; so the typical region that we study has many many galaxies in it.

The fact that we are studying physics on such long scales actually *simplifies* life a lot. To us, the universe seems like it depends on space (e.g. right here, we have the earth, but move a few million km to the left and we have empty space!). However, if you zoom out to cosmological scales, then you start to lose track of these fine-grained inhomgeneities, and the universe starts to look smooth again.

4.1 Kinematics: FRW metrics

To that end, we will make two assumptions about the *spatial* metric of the *universe* on long distance scales:

1. **Isotropy**: this means that if you look around the universe looks the same at all angles, i.e. there is no preferred spatial direction. This is the same as *spherical symmetry*, which we have encountered before.
2. **Homogeneity in** ***space***: this means that the universe is the same at all spatial points.[4]

Philosophically, these ideas go back to Copernicus, and are some times called the **Copernican principle** (after his idea that the earth is, in fact, not the center of the universe). Note that these two ideas are logically distinct; for example, a *cone* is isotropic about its vertex, but clearly not homogenous, whereas a cylinder is homogenous but not isotropic.

Let's now implement these two assumptions mathematically to write down the most general metric that we need to describe the universe. We call the time direction t as usual, and the remaining 3 directions are x^i. Before using the two conditions above, the most general metric is

$$ds^2 = -A(t, x^i)dt^2 + B_i(t, x^i)dtdx^i + h_{ij}(t, x^i)dx^i dx^j \tag{4.1}$$

Isotropy implies that there no preferred spatial direction: but if B_i was nonzero it would define one, so $B_i = 0$. Similarly homogeneity means that A cannot depend on x^i. Thus the term is just $A(t)dt^2$, and by redefining $dt \to \frac{1}{\sqrt{A}}dt$ we can absorb it into the dt^2 term.

Naively, one would conclude that homogeneity means that $h_{ij}(t, x^i)$ would not depend on x^i at all. Actually, as we will see, this is not quite true: instead it means that the dependence on time in $h_{ij}(t, x^i)$ must be an overall factor, so we have the simpler:

$$ds^2 = -dt^2 + a(t)^2\gamma_{ij}(x^i)dx^i dx^j \tag{4.2}$$

where $a(t)$ is called the *scale factor*.

Now let us focus on the spatial 3-metric γ_{ij}. We know that it is isotropic and homogenous. How many homogenous and isotropic 3-manifolds are there? Isotropic (i.e. spherically symmetric) means that if we put

[4] To be more precise: given two spatial points p and q, there is a coordinate transformation that leaves the metric invariant and maps the point p to q. For example in flat space $\mathbb{R}^3$ translations $x^i \to x^i + a^i$ move you all around the manifold, and clearly leave the metric invariant.

down polar coordinates we have

$$d\sigma^2 = \gamma_{ij} dx^i dx^j = \exp(2B(r))dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.3}$$

where the reasoning is the same as in the Schwarzschild case. What are possible choices for $B(r)$? Here we use homogeneity. Recall that we want the metric to be the same everywhere.

We now calculate the 3d Ricci scalar associated to the 3d metric γ_{ij}: a short calculation yields:

$$^{(3)}R = \gamma^{ij}\left({}^{(3)}R_{ij}\right) = \frac{2e^{-2B}}{r^2}\left(-1 + e^{2B} + 2rB'\right) \tag{4.4}$$

Now because the spatial metric is homogenous, the Ricci scalar must be a constant. Next, let's note that if we rescale the spatial metric by a number $\lambda > 0$, then the Ricci tensor and the 3d Ricci scalar transform as

$$\gamma_{ij} \to \lambda\gamma_{ij} \qquad {}^{(3)}R_{ij} \sim \partial\gamma^{-1}\partial\gamma \text{ (unchanged)} \qquad {}^{(3)}R \to \lambda^{-1}\left({}^{(3)}R\right) \tag{4.5}$$

Why am I keeping $\lambda > 0$? If $\lambda < 0$, then the signature of the spatial metric will change, and I don't want to do that.

If we perform such an operation, then we thus rescale the Ricci scalar by a positive number. Note that the overall size of the spatial metric is in the scale factor anyway. Thus we can set the 3d Ricci scalar to one of three values:

$$^{(3)}R = 6\kappa \qquad \kappa = +1, -1, 0 \tag{4.6}$$

Now let us view (4.4) as an equation for $B(r)$. Its general solution is (check!)

$$e^{2B(r)} = \frac{1}{1 - \kappa r^2 - cr^{-1}} \tag{4.7}$$

with c an integration constant. Note however, that if $c \neq 0$ then we have a singularity at $r = 0$; we don't want such a singularity at the origin of polar coordinates, so we set it to 0. Thus at the end of the day, we conclude that the homogenous, isotropic spatial metrics are:

$$d\sigma^2 = \gamma_{ij} dx^i dx^j = \boxed{\frac{dr^2}{1 - \kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)} \qquad \kappa = 0, \pm 1 \tag{4.8}$$

Now we discuss the three cases separately:

1. **Flat space:** $\kappa = 0$: this is just flat space, $\mathbb{R}^3$.
2. **Closed universe:** $\kappa = +1$: this is

$$ds^2 = \frac{dr^2}{1 - r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.9}$$

It turns out this is actually a 3-dimensional sphere S^3. To understand this, consider the coordinate transformation

$$\frac{dr}{\sqrt{1 - r^2}} = d\chi \qquad \to \qquad \sin^{-1}(r) = \chi \tag{4.10}$$

so the metric is then

$$ds^2 = d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.11}$$

From here, we can see that what is happening is that at each point in χ there is a 2-sphere: $\chi \in [0, \pi]$, and the 2-spheres get bigger and then get smaller. This is basically the definition of a 3-sphere. Note that in this case, the universe is *compact*, and thus it is called **closed**.

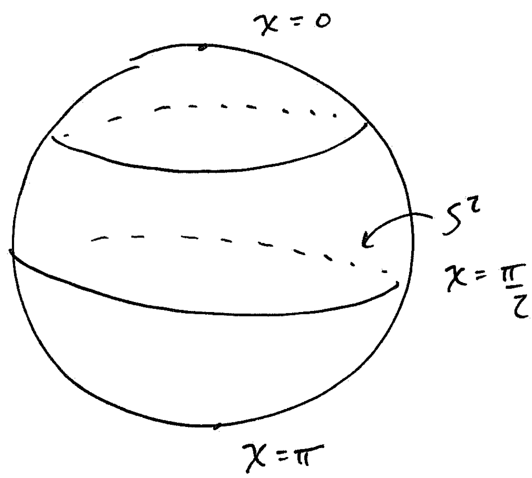

Figure 4.1: An S^3 is a series of S^2's that get bigger and then smaller again.

3. **Open universe:** $\kappa = -1$. In this case the spatial geometry is a negatively curved manifold called the hyperbolic 3-space, or H^3, with metric:

$$ds^2 = \frac{dr^2}{1+r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.12}$$

Note that we can perform a similar coordinate transformation as before

$$\frac{dr}{\sqrt{1+r^2}} = d\chi \qquad \rightarrow \qquad \sinh^{-1}(r) = \chi \tag{4.13}$$

to write the metric as

$$ds^2 = d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.14}$$

Note that unlike the 3-sphere above, as we increase χ, the 2-spheres keep on getting bigger. Thus this universe does not close in on itself, it is *open*.

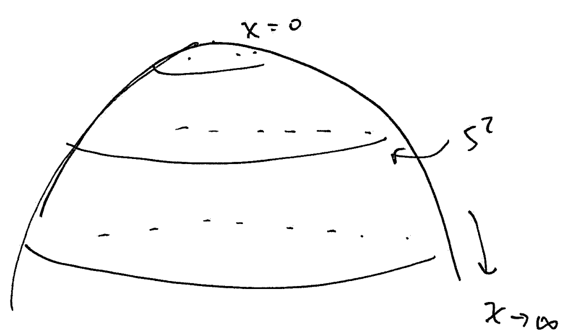

Figure 4.2: An H^3 is a series of S^2's that get bigger exponentially at large proper distance.

There is an important thing to note here: all three of these spatial manifolds are isotropic about every point and homogenous, by construction. They are all equally symmetric, in that they all have just as many Killing vectors as $\mathbb{R}^3$, that is *six*. It turns out this is the maximum possible number of Killing vectors that a 3d space can have: we can say that these spatial manifolds are maximally symmetric in three dimensions. We note that a maximally symmetric space in d dimensions has N_d Killing vectors, where

$$N_d = \frac{d(d+1)}{2} \tag{4.15}$$

(Recall that you showed in a problem that Minkowski space in four dimensions had 10 Killing vectors). These are all possibilities for the spatial sections of our universe.

Putting together all the pieces, we see that the metric now looks like:

$$\boxed{ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right)} \tag{4.16}$$

This is called the Friedmann-Robertson-Walker metric. Clearly if $a(t)$ increases with time, then we will have an expanding universe. Eventually we will use Einstein's equations to determine how $a(t)$ actually does depend on time, but first we will study how observers move in this spacetime.

4.1.1 Observers and redshifts

Let us now discuss the physics of freely falling observers in this spacetime. First, consider an observer at rest at a particular point, i.e. they have

$$\frac{dt}{ds} = 1 \qquad \frac{dx^i}{ds} = 0 \tag{4.17}$$

From direct calculation, we can see that $\Gamma^i_{tt} = 0$: this means that $\frac{d^2x^i}{ds^2} = 0$, and thus $x^i(s) = x^i_0$ is a solution to the geodesic equation. These are called *comoving observers*.

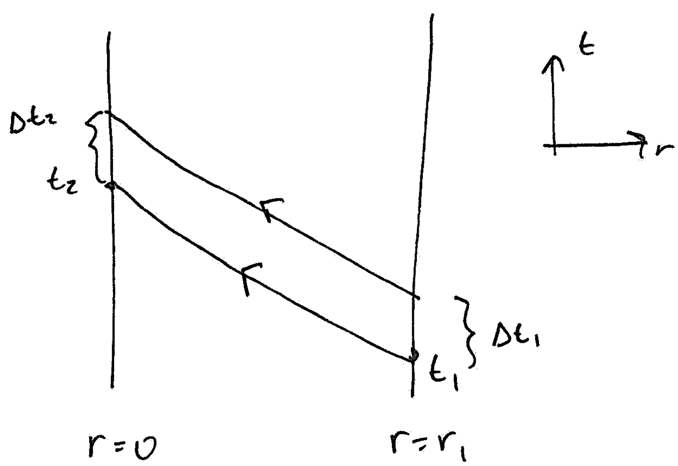

Figure 4.3: Observers used in finding cosmological redshift

Now consider an observer at $r = r_1$ and a different observer (us) at $r = 0$. Suppose the observer at r_1 sends out a light ray at t_1, and that it reaches us at t_2. From the null ray condition we have

$$\dot{t}^2 - a(t)^2\dot{r}^2 = 0 \qquad \frac{dr}{dt} = -\frac{1}{a(t)} \qquad r_1 - 0 = \int_{t_1}^{t_2} dt \frac{1}{a(t)} \tag{4.18}$$

Now suppose he sends out another ray after a time that he measures to be Δt_1 after the first, and it reaches us at a time Δt_2 after the first. Then we find

$$r_1 = \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} \frac{dt}{a(t)} \tag{4.19}$$

Using the fundamental theorem of calculus, we see that

$$\frac{\Delta t_2}{a(t_2)} - \frac{\Delta t_1}{a(t_1)} = 0 \qquad \rightarrow \qquad \frac{\Delta t_2}{\Delta t_1} = \frac{a(t_2)}{a(t_1)} \tag{4.20}$$

Thus, if $a(t_2) > a(t_1)$, then we observe a larger Δt then that from the emitter; this is another example of a redshift. This is called *cosmological redshift*. Astronomers define the cosmological redshift as

$$z = \frac{\Delta t_2}{\Delta t_1} - 1 = \frac{a(t_2) - a(t_1)}{a(t_1)} > 0. \tag{4.21}$$

A larger z means that the source is more redshifted, and also that the universe was a little smaller when the light ray was emitted. One way to think about this is that the universe is expanding, so all other sources are moving away from us – but their movement creates a Doppler shift, which is this redshift.

All of this was general, but now let's consider a source that wasn't too far away, so that the signal was emitted fairly recently, i.e. t_1 is close to t_2. The instantaneous distance between us and the source is then

$$d \approx a(t_1)r_1 \approx (t_2 - t_1) \tag{4.22}$$

where the last equality follows from the nullness condition (4.19). We may now expand the red-shift formula:

$$z \approx \frac{\dot{a}(t_1)(t_2 - t_1)}{a(t_1)} \approx \left.\frac{\dot{a}}{a}\right|_{t_2} d + \mathcal{O}((t_1 - t_2)^2) \tag{4.23}$$

The ratio $\frac{\dot{a}}{a}$ measures how fast the universe is expanding, and is called Hubble's constant H_0:

$$\boxed{H_0 \equiv \frac{\dot{a}(t_1)}{a(t_1)}} \tag{4.24}$$

Thus we may rewrite the red-shift formula as

$$z \approx H_0 d \tag{4.25}$$

The redshift is measurable; it can be understood as measuring how quickly the emitter is moving away from you, and what this means is that if you look out at at the galaxies, you can plot the redshift against how far away they are, and it should form a straight line.

Next, let's make a plot of $a(t)$. It will be increasing, as in Figure 4.4; we see that H_0 measures *very roughly* how long ago the universe had scale factor 0, i.e the age of the universe! Current measurements give

$$H_0^{-1} \approx 14.4 \times 10^9 \text{years} \tag{4.26}$$

This is an overestimate; the actual age is 13.7 billion years.

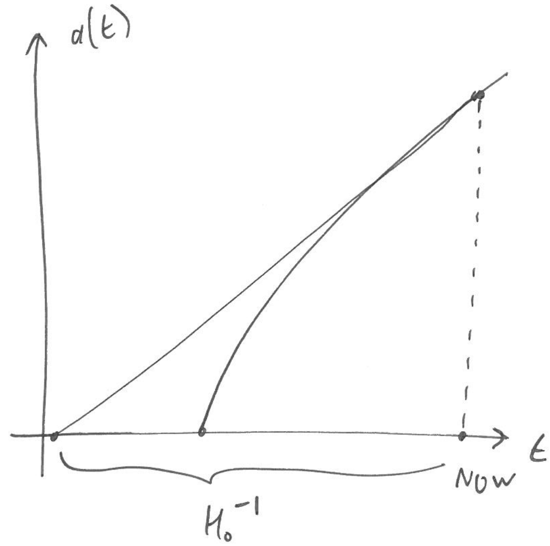

Figure 4.4: Estimating the age of the universe using the Hubble parameter H_0 now.

4.2 Dynamics: the Friedmann equations

We now seek to understand how $a(t)$ evolves in time. To do this, we study the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{4.27}$$

where $T_{\mu\nu}$ is the stress energy tensor for all of the matter that makes up the universe. It is also constrained by isotropy and homogeneity to take the *perfect fluid* form:

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu} \qquad U^\mu = \delta^\mu_t \tag{4.28}$$

Note that it is particularly nice to write it with "one up, one down" indices, in which case we have

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \tag{4.29}$$

Here ρ and p are the energy and pressure of the fluid that we are talking about. It turns out for any fluid they are related by some function $p = p(\rho)$; this function is called the *equation of state*, and it depends on which fluid you are talking about in particular. In all the cases that we are interested in, the equation of state always takes the form

$$p = w\rho \tag{4.30}$$

where w is a constant (in fact, a dimensionless number). Now we discuss some of the simple cases:

1. **Matter:** this is ordinary *stuff*, e.g. galaxies, stars, etc. – it is often called "dust". It has zero pressure:

$$p = 0 \qquad w_{matter} = 0 \tag{4.31}$$

   Later on we will discuss the composition of the matter of our actual universe; it turns out we only understand a very small part of what makes up our universe.

2. **Radiation:** this is electromagnetic radiation. We have previously looked at the stress tensor for the electromagnetic field. It turns out (look back at your notes!) that the trace of the stress tensor $T^{\mu}{}_{\mu} = 0$; we see from the above parametrization that this means that

$$3p = \rho \qquad w_{radiation} = \frac{1}{3} \tag{4.32}$$

This was very important in the early universe.

3. **Vaccuum energy**: the final case is called "vaccuum energy": for this case we have

$$p = -\rho \qquad w_{vacuum} = -1 \tag{4.33}$$

In this case the stress tensor is proportional to the metric itself. This is the case if have a *cosmological constant*, which you will recall from (2.64) affected the Einstein equations like

$$T^{(\Lambda)}_{\mu\nu} = -\frac{1}{8\pi G}\Lambda g_{\mu\nu} \tag{4.34}$$

Sometimes this contribution is taken out and written separately from the other terms for pressure and energy. In our universe this contribution is eventually very important.

Now that we understand the different possible sorts of matter, we study the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{4.35}$$

Now we need to perform some algebra. We plug in the FRW metric (4.16) into the Einstein equations and work out the different components of $R_{\mu\nu}$. Again, I do not work out the details. After some algebra we find that the tt component of the above equation is:

$$3\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right) = 8\pi G\rho, \tag{4.36}$$

We divide it by 3 to get the equation in the form that it is usually written, called the *Friedmann equation*:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho} \tag{4.37}$$

All of the ij equations are proportional to each other because of isotropy. The $\theta\theta$ equation is:

$$-r^2\left(\kappa + \dot{a}^2 + 2a\ddot{a}\right) = 8\pi G a^2 p r^2 \tag{4.38}$$

Now we can eliminate κ from these to find the second equation, called the Rayachaudhuri equation:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)} \tag{4.39}$$

These are the two fundamental equations that govern cosmological dynamics. They contain all the information needed to solve for the evolution of the universe. There is however a useful manipulation that we can perform: consider taking a time derivative of the Friedmann equation and solving for $\ddot{a}$ to find

$$\frac{\ddot{a}}{a} = \frac{\kappa}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{4\pi G}{3}\frac{a\dot{\rho}}{\dot{a}} \tag{4.40}$$

Combining this with the Friedmann equation to eliminate $\ddot{a}$, we find eventually the following simple equation:

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0} \tag{4.41}$$

As it turns out, this equation is actually equivalent to $\nabla_\mu T^{\mu\nu} = 0$, i.e. the conservation of the stress tensor, which is something that you will verify in a problem set. This is thus sometimes called the conservation equation.

4.3 Cosmological solutions

Now we will seek to understand the solutions to the equations of motion. Note that these three equations are not all independent, so we really need only two of them to get a closed system. Usually we use (4.41) to figure out the evolution of the matter with time; then we can plug that back into (4.37) to get a closed equation.

4.3.1 Dilution of matter and radiation

To understand this, let's first look at (4.41) for the different cases. Note that as $p = w\rho$, we have

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1+w)\rho = 0 \to \frac{d}{dt}(\rho a^{3(1+w)}) = 0 \tag{4.42}$$

which means that for all cases we have

$$\rho \sim \frac{\rho_0}{a^{3(1+w)}} \tag{4.43}$$

So now let's understand what this means in all cases:

1. **Matter.** Here $w = 0$; thus we have

$$\rho_{matter}(a) = \frac{\rho_0}{a^3} \tag{4.44}$$

Thus the energy density in the matter decreases like the cube of the scale factor. This can be understood as saying that there is a bunch of matter in the box, but the number of particles stays the same even though the box is getting bigger. The energy per particle stays the same, so the density gets smaller.

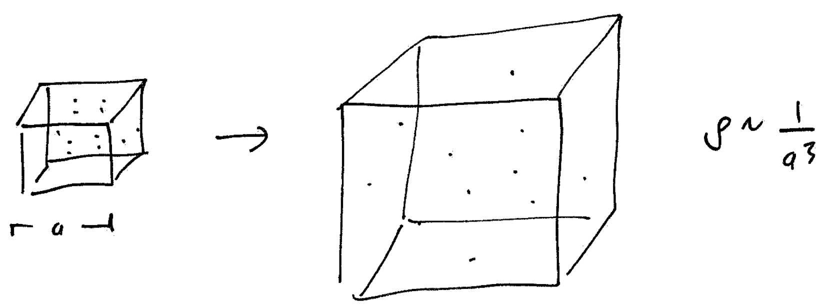

Figure 4.5: Dilution of matter as universe expands

2. **Radiation:** Here $w = \frac{1}{3}$, thus we have

$$\rho_{radiation}(a) = \frac{\rho_0}{a^4} \tag{4.45}$$

This is similar. The difference now is that the energy of a single photon goes like $\hbar\lambda^{-1}$, where λ is the wavelength of the photon; so in addition to the suppression for matter, there is extra suppression from the fact that the box is stretching. This results in the extra suppression by a factor of a.

3. **Vacuum energy:** Here $w = -1$, so we have

$$\rho_{vacuum} \sim \text{const} \tag{4.46}$$

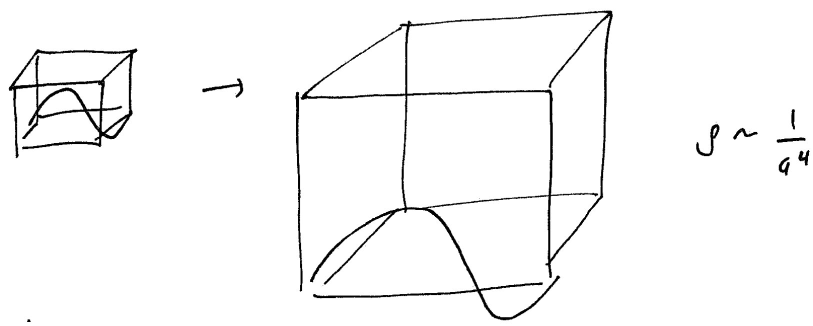

Figure 4.6: Dilution of radiation as universe expands: note extra dilution due to stretching of typical wavelength as box gets bigger.

This is weird. Vacuum energy (i.e. a cosmological constant) does not decrease with time, it stays the same as the universe expands. In some sense it is a property of the space itself, not of anything else living on the space. This is so peculiar that it is conventional to separate out its contribution and write the Friedmann equation as

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho(a) + \frac{\Lambda}{3} \tag{4.47}$$

where it is understood in this equation that ρ represents all matter-energy that is not from Λ (and thus depends on a in some manner), but that Λ is a constant.

4.3.2 Actual solutions

Now it is straightforward to solve this system of equations. In practice, since the different kinds of stress energy fall off at different rates, at any one time only *one* kind is important. So we will treat them separately.

Matter or radiation domination

Let's start by setting $\Lambda = 0$, and looking at either only matter or radiation. We will treat them together; plugging the known dependence of ρ on a into the Friedmann equation, we find

$$\rho \sim \frac{\rho_0}{a^3} \qquad \rightarrow \qquad \dot{a}^2 - \frac{8\pi G}{3}\frac{\rho_0}{a} = -\kappa \tag{4.48}$$

$$\rho \sim \frac{\rho_0}{a^4} \qquad \rightarrow \qquad \dot{a}^2 - \frac{8\pi G}{3}\frac{\rho_0}{a^2} = -\kappa \tag{4.49}$$

where the left case is for matter domination, and the right-hand side is for radiation domination. We can use our usual trick; this looks like an equation for energy conservation of a particle with coordinate $a(t)$, potential $V(a) = -\frac{8\pi G}{3}\frac{\rho_0}{a^{1,2}}$, and energy $\kappa = \pm 1, 0$.

Note that if $\kappa = 0$ or -1 the universe expands forever, whereas if $\kappa = +1$ then the universe expands up to a maximum size and then falls back again (Big Crunch). It is entirely possible to work out the time dependence of all of these. I'll only do it for the $\kappa = 0$ case. For matter domination we have:

$$\dot{a}^2 = \frac{\alpha}{a} \qquad \rightarrow \qquad \boxed{a_{matter}(t) = c_1(t - t_0)^{\frac{2}{3}}} \tag{4.50}$$

and for radiation domination we have

$$\dot{a}^2 = \frac{\beta}{a^2} \qquad \rightarrow \qquad \boxed{a_{radiation}(t) = c_2\sqrt{t - t_0}} \tag{4.51}$$

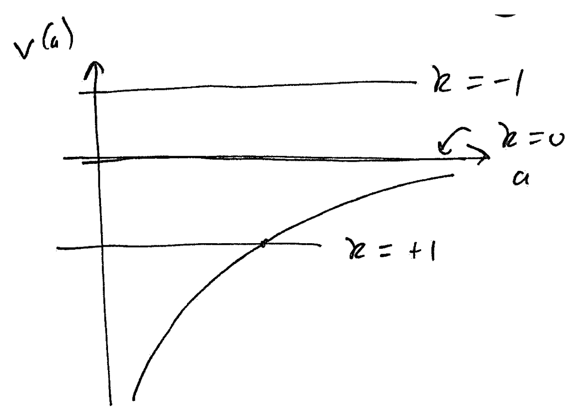

Figure 4.7: Effective potential $V(a)$ for cosmological dynamics.

$c_{1,2}$ are constants that can be related to α, β, but doing so is not very illuminating, so I won't. Note that the universe expands, and we can figure out how fast it does so.

Vacuum Domination

Let's now look at vacuum domination, i.e. we assume the full stress tensor is given by the cosmological constant:

$$\dot{a}^2 + \kappa = \frac{\Lambda}{3} a^2 \tag{4.52}$$

Let's assume $\Lambda > 0$. Then we find the following three solutions for each value of κ:

$$\kappa = 0 : \qquad a(t) = a_0 e^{\pm \frac{t}{\ell}} \tag{4.53}$$

$$\kappa = 1 : \qquad a(t) = \ell \cosh\left(\frac{t}{\ell}\right) \tag{4.54}$$

$$\kappa = -1 : \qquad a(t) = \ell \sinh\left(\frac{t}{\ell}\right) \tag{4.55}$$

where in all cases $\ell = \sqrt{\frac{3}{\Lambda}}$.

These solutions are important for a few different reasons. Note that they grow *exponentially* in time rather than polynomially; somehow these universes expand faster.

Next, though this is far from clear, they actually all represent the same space time written in different coordinates: this spacetime is called de-Sitter space, and it is maximally symmetric in 4 dimensions (not 3!). Thus it has 10 Killing vectors, and is of intrinsic interest as a symmetric solution to general relativity.

Secondly: if you have any spacetime with some matter and radiation but even a little bit of cosmological constant, then the matter and radiation will eventually dilute down to zero, but the cosmological constant will survive, and eventually our universe will be described by de Sitter space.

As an aside, I mention that there is such a thing called *Anti* de Sitter space; it is what you get if you have $\Lambda < 0$. It is the favorite spacetime of many string theorists, including myself.

4.4 Our own universe

Now we have some understanding about how different sorts of matter cause a hypothetical universe to evolve. Let us turn to our own universe. To do this, we first introduce some terminology. If we allow matter, radiation, and cosmological constant to *all* be present, then we find that the Friedmann equation is

$$\frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\left(\rho_{matter} + \rho_{rad}\right) + \frac{\Lambda}{3} \tag{4.56}$$

where I have explicitly separated all of the components, and where we know that $\rho_{matter} \sim a^{-3}, \rho_{rad} \sim a^{-4}$, and Λ does not change in time. Recall now also that we have the Hubble "constant"

$$H \equiv \frac{\dot{a}}{a} \tag{4.57}$$

It is now convenient to divide the whole thing by H^2. We get

$$1 + \frac{\kappa}{H^2 a^2} = \frac{8\pi G}{3H^2}\left(\rho_{matter} + \rho_{rad}\right) + \frac{\Lambda}{3H^2} \tag{4.58}$$

Now we introduce some terminology: we see that there are a bunch of things that add up to 1; thus all of these things must be dimensionless. We define

$$\Omega_c \equiv \frac{\kappa}{H^2 a^2} \tag{4.59}$$

$$\Omega_M \equiv \frac{8\pi G}{3H^2}\rho_{matter} \tag{4.60}$$

$$\Omega_R \equiv \frac{8\pi G}{3H^2}\rho_{rad} \tag{4.61}$$

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H^2} \tag{4.62}$$

and thus we can rewrite the equation as

$$1 + \Omega_c = \Omega_M + \Omega_R + \Omega_\Lambda \tag{4.63}$$

Note that Ω_c has to do with spatial curvature, whereas $\Omega_{M,R,\Lambda}$ are proportional to the contribution of matter, radiation and vacuum energy to the energy budget of the universe respectively. It is conventional to define one more thing, called Ω: the *density parameter*, to be the sum of everything on the right hand side:

$$\Omega \equiv \Omega_M + \Omega_R + \Omega_\Lambda \tag{4.64}$$

and so we have the cute equation

$$\boxed{1 + \Omega_c = \Omega} \tag{4.65}$$

Ω measures the total amount of energy that is present in the universe: note that if $\Omega > 1$, then we know that $\Omega_c > 0$, and thus we have $\kappa = 1$; similarly for a flat universe we require $\Omega = 1$ (as $\Omega_c = 0$) and for an open universe we have $\Omega < 1$. Thus Ω tells us that if we have too much energy density in the universe, then the universe collapses into a ball and becomes compact. For that reason one sometimes writes Ω as

$$\Omega = \frac{\rho_{tot}}{\rho_{crit}} \qquad \rho_{crit} = \frac{3H^2}{8\pi G} \tag{4.66}$$

with ρ_{crit} called the *critical density* and ρ_{tot} counting all the contributions (matter, radiation, vacuum).

Ok, I think I have now introduced all the relevant terminology. Note that all of these things change with time as the densities change. We can now discuss our own universe.

4.4.1 The energy budget

Direct measurements of the gravitational effects of the matter content of the universe at this moment reveal that

$$\Omega_{M0} = 0.3 \pm 0.1 \tag{4.67}$$

where the 0 subscript indicates that we are talking about right *now*. Now let us talk about this matter: what is it made of? If you look at matter around you, it's made of protons, neutrons, electrons, etc. We have a very good understanding of such ordinary matter. However if we study the contribution of such "ordinary matter", we find that the ordinary matter contributes only

$$\Omega_{M0,\text{ordinary}} = 0.04 \pm 0.02 \tag{4.68}$$

So what is the rest? We don't know – but we know that it does not give off any ordinary light that we can see with our eyes (or our telescopes, etc.), so it is called *dark matter*. It is most likely an entirely new kind of particle that we have not yet detected.

Next up, what about the radiation contribution? As we know, the radiation falls off faster with time than the matter, so we expect it to contribute less; we expect it to be around

$$\Omega_{R0} \sim 10^{-4} \tag{4.69}$$

So for all intents and purposes, we may ignore it.

Next, we turn to spatial curvature. It appears that it is very close to zero, i.e .

$$|\Omega_{c0}| < 0.1 \tag{4.70}$$

So if someone asks you the shape of the universe, you can say that (within error bars), it is flat.

So that leads us with vacuum energy, also called *dark energy*. Direct measurements from the redshifts of supernovae – which is nicely consistent with the equations above, gives us

$$\Omega_{\Lambda 0} = 0.7 \pm 0.1 \tag{4.71}$$

Theoretically, we simply do *not* understand what this energy is. A brief digression: this is energy associated with the *vacuum*; if you believe in quantum mechanics, then there is something called the *zero-point energy* associated with the fact that particles and anti-particles can spontaneously appear and disappear into the vacuum. This should contribute to the energy of the vacuum, but if you try to add up those energies, you do get a number that is 10^{123} bigger than the observed one:

$$\rho_{\Lambda 0,\text{theory}} \sim 10^{123} \times \rho_{\Lambda 0,\text{experiment}} \tag{4.72}$$

So this is a terrible disaster from a theoretical point of view, and is called the *cosmological constant problem.* Some view this as perhaps the single most confusing problem in theoretical physics today.

So in a nutshell, then: 70% of the universe is vacuum energy, and we don't understand what it is. Of the 30% left, 25% or so is dark matter, and we don't understand what that is. 5% or so is ordinary matter – we understand that.

Clearly, there is work to do.

4.4.2 History

Having exhaustively discussed the universe *now*, we now turn to its history.

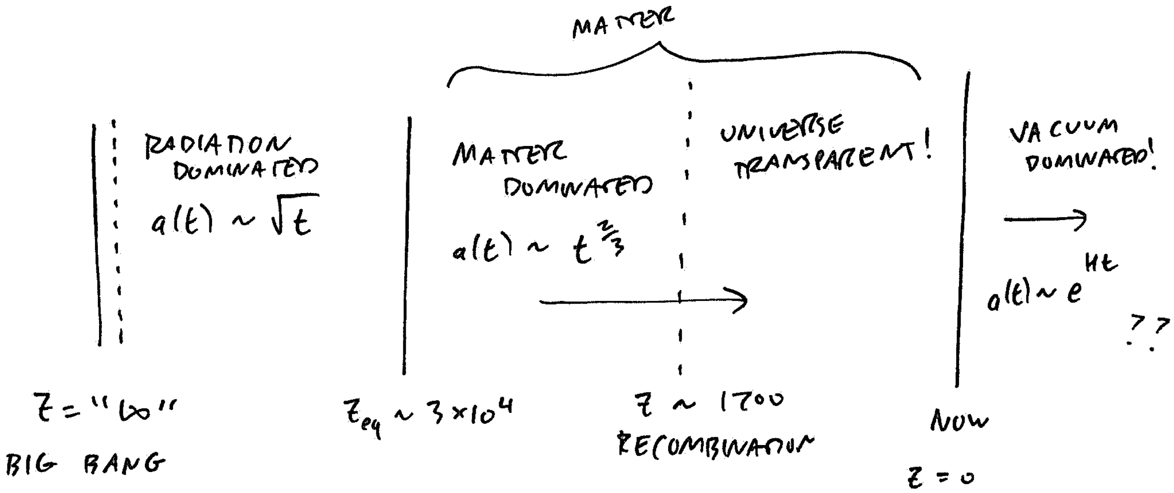

Figure 4.8: Rough timeline of the universe

The universe began about 13.7 billion years ago. Rather than discuss times, I will talk about times in terms of redshift z. Recall that the redshift of a photon emitted at a time t_1 and observed now was defined as

$$z(t_1) = \frac{a(t_{\text{now}}) - a(t_1)}{a(t_1)} \tag{4.73}$$

So $z(t_{\text{now}}) = 0$, and z gets bigger as we go deep into the past.

Now the universe is full of radiation and matter; for most intents and purposes the radiation behaves like it is a gas with a temperature T. This temperature cools down as the universe expands as:

$$T(t) \sim \frac{1}{a(t)} \tag{4.74}$$

This is basically the same information as the radiation formula $\rho_{rad} \sim a^{-4}$, but to explain the precise relation requires a little bit of statistical mechanics.

Let's say the Big Bang happened at some time t_{BB} when $a(t_{BB}) = 0$. This is a curvature singularity of the metric; we cannot understand much about this time. Note from (4.74) we see that the temperature $T(t_{BB})$ at that point is technically *infinity*; thus some people say that the Big Bang is an explosion. Note that people also say that the universe began from a point; this is not really true, as it may have been flat (and thus goes on forever), but it is also infinitely small. We also have $z(t_{BB}) \sim \infty$. We will start our considerations at some time slightly later than the Big Bang; here everything is very high but still understandable.

Meanwhile the universe starts expanding. The temperature cools down, and eventually we begin to understand the physics. There is a gas of particles: importantly, the temperatures are so high that electrons and protons move around freely: they have so much kinetic energy that they do not know that they attract each other. Photons bounce around and scatter off of these charged particles. At early times the radiation is more important, and so the universe expands as $a(t) \sim \sqrt{t}$. But the radiation density falls as $\rho_{rad} \sim a^{-4}$ whereas $\rho_{matter} \sim a^{-3}$, so eventually the matter catches up! This turns out to happen at

$$z_{\text{eq}} \approx 3 \times 10^4 \tag{4.75}$$

so still pretty far in the past. From then on the universe is matter dominated, so we have $a(t) \sim t^{\frac{2}{3}}$.

Another critical moment happens a bit later: note that as the temperature cools, eventually protons and electrons bond together and form hydrogen molecules. At this instant, suddenly there are no charged particles floating around any more: this has a profound effect, as now the photons can go through them freely. At this instant, the universe goes transparent: this is called *recombination*, and it happens at:

$$z_{\text{recombination}} \approx 1200 \tag{4.76}$$

After this life goes on, galaxies form, etc. etc. until very close to the present, until the dark energy starts to become important. At the present moment the matter and dark energy are pretty comparable, as we discussed previously.

Note that the radiation that I talked about is still present in the sky: its temperature has become very small, but it is called the cosmic microwave background, or CMB. The temperature is

$$T_{CMB} \approx 2.7K \tag{4.77}$$

and you can see it if you look out into the sky. Here is a map: we are actually looking at radiation from recombination.

Later on in the future, the dark energy will be more and more important, as the matter redshifts away. Thus at late times we will have

$$a(t \to \infty) \sim e^{\ell t} \tag{4.78}$$

with the universe expanding exponentially in a de Sitter phase.

4.5 A few issues

This is our current understanding of the universe. Now I want to point out a few (more) issues with it that should encourage you to realize that the story I just told you is in fact *incomplete*. These mostly have to do with *initial conditions* and how weird they are.

The flatness problem

Recall from (4.70) that the observed value for the curvature parameter today Ω_{c0} is very small. I want to now ask whether or not this makes *sense*. To understand this, let us derive an equation for how the curvature parameter varies with time, i.e. I seek a formula for $\dot{\Omega}_c$, in terms of the other dimensionless objects Ω_i. This will also give us some practice in manipulating the equations of motion.

We begin by writing down everything. First the Friedmann and Rayachaudhuri equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho_{tot} \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_{tot} + 3p_{tot}) \tag{4.79}$$

where ρ_{tot} and p_{tot} is a sum of the contributions from matter, radiation, and vacuum energy. Each component satisfies the independent equation of state

$$p_i = w_i \rho_i \tag{4.80}$$

where i runs over matter, radiation and vacuum:

$$w_M = 0 \qquad w_R = \frac{1}{3} \qquad w_\Lambda = -1 \tag{4.81}$$

Finally we recall the definitions of the density parameters:

$$\Omega_c = \frac{\kappa}{H^2 a^2} \qquad \Omega_M = \frac{8\pi G}{3H^2}\rho_M \qquad \Omega_R = \frac{8\pi G}{3H^2}\rho_R \qquad \Omega_\Lambda = \frac{8\pi G}{3H^2}\rho_\Lambda \tag{4.82}$$

Now we begin. To derive an equation for the time derivative of Ω_c, we just directly compute:

$$\dot{\Omega}_c = -\frac{2\kappa}{H^3 a^2}\dot{H} - \frac{2\kappa}{H^2 a^3}\dot{a} = -2\Omega_c\left(\frac{\dot{H}}{H} + \frac{\dot{a}}{a}\right) \tag{4.83}$$

Now we want an expression for $\dot{H}$. From the definition $H = \dot{a}/a$ we find that

$$\frac{\dot{H}}{H^2} = \frac{1}{H^2}\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right) \tag{4.84}$$

Now using the Rayachaudhuri equation we eliminate $\ddot{a}a$ to find

$$\frac{\dot{H}}{H^2} = \frac{1}{H^2}\left(-\frac{4\pi G}{3}(\rho_{tot} + 3p_{tot}) - H^2\right) \tag{4.85}$$

Using the equation of state we replace each pressure with its corresponding energy density: we thus find

$$\frac{\dot{H}}{H^2} = \frac{1}{H^2}\left(-\frac{4\pi G}{3}\sum_i \rho_i(1+3w_i) - H^2\right) \tag{4.86}$$

where the sum in i runs over radiation, matter, and vacuum as in (4.80). We now use the definitions of the density parameters to find

$$\frac{\dot{H}}{H^2} = -\frac{1}{2}\sum_i (1+3w_i)\,\Omega_i - 1 \tag{4.87}$$

Now inserting this back into (4.83) we find

$$\dot{\Omega}_c = -2\Omega_c H \left(-\frac{1}{2}\sum_i (1+3w_i)\,\Omega_i\right) \tag{4.88}$$

Finally putting in the numerical values for the w_i from (4.81) we get

$$\boxed{\dot{\Omega}_c = \Omega_c H \left(\Omega_M + 2\Omega_R - 2\Omega_\Lambda\right)} \tag{4.89}$$

This is rather a nice expression, as it removes all of the ugly dimensionful constants that are present in the equations for cosmological evolution: instead it simply directly tells us how all of the dimensionless density parameters evolve with time.

Now let us apply this to the very early universe, i.e. right after the big bang. From our earlier discussion, we see that $\Omega_\Lambda \approx 0$ then, i.e. vacuum energy was essentially negligible. Thus we have

$$\dot{\Omega}_c = \Omega_c H \left(\Omega_M + 2\Omega_R\right) \tag{4.90}$$

Now, what sort of equation is this for Ω_c? Note that if $\Omega_c = 0$, then it stays zero for all time; thus the situation with zero curvature is a solution.

But what if we have a small nonzero Ω_c? Since both Ω_M and Ω_R are *positive*, a small nonzero positive Ω_c will get bigger and bigger with time, and a small nonzero negative Ω_c will get more and more negative with time. Thus we see that $\Omega_c = 0$ is an *unstable* point.

So: does it then make sense for Ω_c to be very small today? Not really! It suggests that Ω_c was somehow fine-tuned to be extremely extremely small (or zero) at the Big Bang. Which is, perhaps, odd. (Or is it? Discuss.)

The Horizon Problem

Now I want to finally conclude by discussing what you see if you look out into space. Consider the following (flat) FRW metric:

$$ds^2 = -dt^2 + a(t)^2\left(dr^2 + r^2 d\Omega^2\right) \tag{4.91}$$

Now let us perform the following redefinition to a new coordinate η:

$$dt = a(t)d\eta \tag{4.92}$$

in terms of which we have

$$ds^2 = a^2(t)\left(-d\eta^2 + dr^2 + r^2 d\Omega^2\right) \tag{4.93}$$

The metric is equal to a scalar function multiplying a flat metric! This scalar function (in this case $a(t)^2$) is sometimes called a *conformal factor*, and since η puts the metric into this "conformal" form, it is called *conformal time*. The point of conformal form is of course that light rays are simple to understand in conformal form: radial null light days simply satisfy

$$\eta = \pm r \tag{4.94}$$

However there is something interesting about conformal time. Let us recall that in our universe at very early times we had radiation domination. Let's pick time coordinates so that the big bang happened at $t = 0$, so

$$a(t) = a_0\sqrt{t} \qquad \frac{d\eta}{dt} = \frac{1}{a(t)} = \frac{1}{a_0 t^{\frac{1}{2}}} \tag{4.95}$$

Now this means that we have

$$\eta(t) = \frac{2}{a_0}\sqrt{t} \tag{4.96}$$

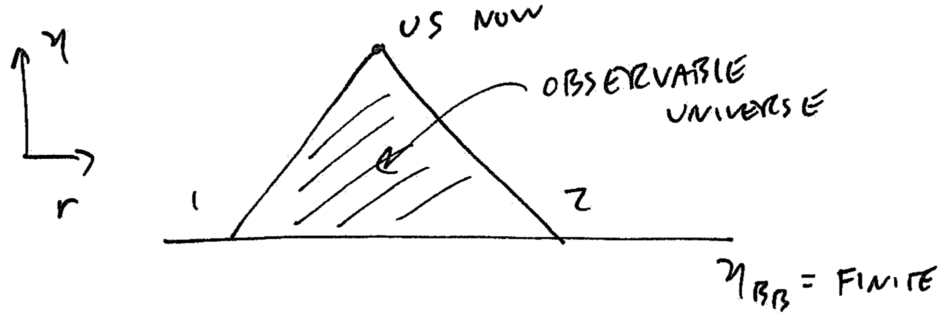

Figure 4.9: Light cone of what we can perceive in conformal time

The important point here is that, at $t = 0$, i.e. at the beginning of the universe, the value of conformal time is *finite*. So imagine that we draw a picture of the universe, using conformal time and r. It looks like:

Now imagine that we sit now and look out into the sky. As we look back, we also look back in time; eventually we look all the way back into the past. Notice that anything outside the light cone, we *cannot see*; there is a finite volume that is accessible to us, and this is called *the Hubble volume.* This is a bit like the event horizon of a black hole.

Now I will pause to indicate a paradox. Imagine that I look at two points at two opposite ends of the sky; on the picture these two points map to diametrically opposite points. Now, these two points *have never been in causal contact.* Yet if we measure the temperature from those two points, the temperatures are *almost the same.* This is peculiar: how did they know that they had to be at the same temperature? Again, this is the same ideas as the curvature problem – it suggests that when the universe was started, God took great care to start all different parts of it at precisely the same temperature: the Big Bang is very finely tuned.

This oddity about the initial conditions of the universe is called the *horizon problem.* (I have simplified it slightly for ease of presentation).

Both of these issues can be solved through the theory of *inflation*; we don't have time to discuss it, but basically the idea is that the universe was de Sitter at very early times as well. This is a fascinating subject that you are now all entirely equipped to study, and I refer you to Sean Carroll's textbook Chapter 8.8 to learn more about it.

5 Gravitational waves

We turn now to the final chapter of this course: gravitational waves. Until recently, this was a purely theoretical subject, but no more.

5.1 Gauge transformations

Before plunging into a study of gravitational waves, we first need to understand exactly what parts of the metric are *physical* and what are not. To do this, let us first start with an example. Consider flat 2d space:

$$ds^2 = dx^2 + dy^2 \tag{5.1}$$

Now consider performing a coordinate change to $x = x' + \epsilon f(y)$, where ϵ is small. We find then the new metric (to lowest order in ϵ):

$$ds^2 = (dx' + \epsilon f'(y)dy)^2 + dy^2 = dx'^2 + 2\epsilon f'(y)dx'dy + dy^2 \tag{5.2}$$

Finally, let us rename x' back to x, which we are always able to do, as it is just a label. We are left with:

$$ds^2 = dx^2 + 2\epsilon f'(y)dxdy + dy^2 \tag{5.3}$$

Now, is this a different geometry than (5.1)? No, it is clearly still flat space, this time expressed in funny coordinates.

However, it is of course a different *metric*: the new metric has this off-diagonal term in it, $g_{xy} \neq 0$. Thus we conclude that two different metrics map to the same spacetime: for historical reasons, this idea is called "gauge invariance". (A similar concept also appears in electrodynamics, where it is a bit more abstract.)

This redundancy is precisely the freedom to choose coordinates (indeed, you will remember that we have exploited this freedom many times in the course so far). We should imagine that there is an abstract manifold $\mathcal{M}$, and there are two different ways to pick coordinates on it, as in Figure 5.1.

Let us now be a bit more precise about this. Consider an infinitesimal change of coordinates: this can be thought of as defining a vector field ζ^μ on space. Roughly speaking, this can be understood as the operation:

$$x'^\mu = x^\mu + \zeta^\mu \tag{5.4}$$

The above equation is, however, somewhat imprecise, as technically speaking you should not add a vector to a coordinate; however hopefully it should be clear what physical idea this is meant to represent.

Nevertheless the physical question remains: how, then, do physical quantities such as the metric (or anything) change under such an infinitesimal coordinate transformation?

We are *tempted* to imagine something like the following: if we had a scalar field, then we could imagine "evaluating the field at the new point", i.e. **put this in quotes**

$$\phi(x') = \phi(x + \zeta) = \phi(x) + \zeta^\mu \partial_\mu \phi \tag{5.5}$$

This is heuristic, but it suggests that the transformation that we are interested **second term** should involve derivatives of the field.

Having motivated it, we now develop the correct mathematical machinery for this, which involves Lie derivatives. These were introduced in the first term (see chapter 11 of the first term lecture notes), but we recall what the definition is. For any geometric object, we can define its *Lie derivative* along a vector field ζ.

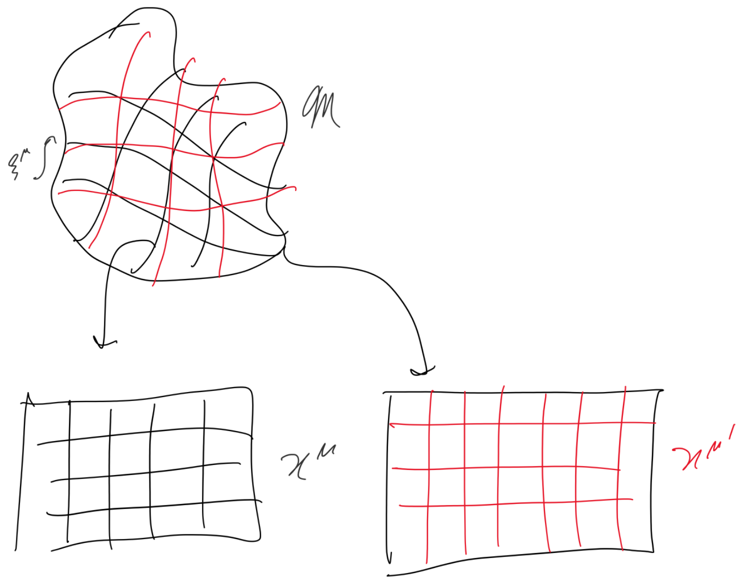

Figure 5.1: Idea of different coordinate systems on a manifold $\mathcal{M}$

There are expressions for the Lie derivatives of objects with any number of up or down indices. For a scalar field this is simply the normal derivative:

$$\mathcal{L}_\zeta \phi = \zeta^\mu \nabla_\mu \phi \tag{5.6}$$

We will particularly care about the case when we have two lower indices: for the Lie derivative of a tensor $A_{\mu\nu}$ along a vector field ζ we have:

$$\mathcal{L}_\zeta A_{\mu\nu} = \zeta^\sigma \nabla_\sigma A_{\mu\nu} + (\nabla_\mu \zeta^\sigma) A_{\sigma\nu} + (\nabla_\nu \zeta^\sigma) A_{\mu\sigma} \tag{5.7}$$

Roughly speaking, this tells us how all of these objects change if we "drag" them along a vector field. Note that when we apply this to the metric we have:

$$\mathcal{L}_\zeta g_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu \tag{5.8}$$

as the covariant derivatives of the metric are zero.

Now let me make a definition. For every vector field ζ^μ, we define a *gauge transformation* of the metric as the infinitesimal transformation:

$$g_{\mu\nu} \to g_{\mu\nu} + \delta_\zeta g_{\mu\nu} \qquad \boxed{\delta_\zeta g_{\mu\nu} = \mathcal{L}_\zeta g_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu} \tag{5.9}$$

Such a transformation is also called a *diffeomorphism*; it can be *morally* understood as an infinitesimal change of coordinates, as in (5.4). Note that the example (5.3) that we started with takes this form, with $\zeta_\mu = 2\epsilon(f(y), 0)$.

The important thing is: two metrics that are related by a gauge transformation are *physically equivalent*; they represent the same geometry, though they are written in different coordinates.

One would now hope that the physics is *invariant* under gauge transformations. This means that if two metrics are related by a gauge transformation:

$$g'_{\mu\nu} = g_{\mu\nu} + \delta_\zeta g_{\mu\nu} \tag{5.10}$$

then we get the same physical answers for all questions we can ask. For example, if $g_{\mu\nu}$ satisfies Einstein's equations in vacuum, then so will $g_{\mu\nu}$:

$$R_{\mu\nu}[g] = 0 \qquad \leftrightarrow \qquad R_{\mu\nu}[g'] = 0 \tag{5.11}$$

where $R_{\mu\nu}[g]$ refers to the Ricci tensor evaluated on the metric g, etc. This can be checked explicitly in full generality, though we will only do the linearized version of it below. It is instructive to check (exercise!) that the *integrated* Einstein-Hilbert action is also invariant under gauge transformations.

How many independent gauge transformations are there? We see that as there are four independent coordinates, there are *four* different functions of space that parametrize gauge transformations in general relativity.

All of this is important conceptually. It is also very useful at a practical level, because it means that we can typically use these gauge transformations to put our metric into a form that simplifies computations.

5.2 Gravitational waves and their polarizations

Armed with this, we may now move on to demonstrate that general relativity supports wave-like solutions that propagate through space.

5.2.1 Linearized Einstein equations

We will consider small ripples around flat space, i.e. we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad h_{\mu\nu} \ll 1 \tag{5.12}$$

where as usual $h_{\mu\nu}$ is a small perturbation and we work only to first order in it. Let's get a feeling for how this works by computing the inverse metric in powers of h. It is fastest to just write down the answer:

$$g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\sigma}\eta^{\nu\rho}h_{\sigma\rho} + \mathcal{O}(h^2) \tag{5.13}$$

and you can check this indeed satisfies $g^{\mu\nu}g_{\rho\sigma} = \delta^{\mu}_{\sigma} + \mathcal{O}(h^2)$, so is correct to this order.

Note that this means that if we raise and lower indices on h itself we can do it with the flat space metric:

$$g^{\mu\nu}h_{\nu\sigma} = \eta^{\mu\nu}h_{\nu\sigma} + \mathcal{O}(h^2) \tag{5.14}$$

In the remainder of this section, every computation will involve a power of h; thus we will raise and lower with just $\eta^{\mu\nu}$.

Finally, note that from the formula for the gauge transformation of $g_{\mu\nu}$ (5.9), the gauge transformation of $h_{\mu\nu}$ is given by

$$\delta_{\zeta}h_{\mu\nu} = \partial_{\mu}\zeta_{\nu} + \partial_{\nu}\zeta_{\mu} \tag{5.15}$$

with ζ an arbitrary 4-vector, where to the lowest order in h we can replace covariant derivatives with partials.

Next, we want to work out the full Einstein equations as a function of this small perturbation h. (Recall at the beginning of term, we did it in the static limit.) To do this, first consider the general expression for the Christoffel symbols

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}\left(\partial_{\mu}g_{\nu\lambda} + \partial_{\nu}g_{\lambda\mu} - \partial_{\lambda}g_{\mu\nu}\right) \ . \tag{5.16}$$

Now, seeing as $\eta_{\mu\nu}$ does not depend on space, to lowest order we can rewrite this as

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\eta^{\sigma\lambda}\left(\partial_{\mu}h_{\nu\lambda} + \partial_{\nu}h_{\lambda\mu} - \partial_{\lambda}h_{\mu\nu}\right) \tag{5.17}$$

Now the Riemann tensor in general is

$$R^{\lambda}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} \ . \tag{5.18}$$

As $\Gamma \sim h$, we may neglect the terms that are Γ^2; we are then left with

$$R^{\lambda}{}_{\rho\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\rho h^{\lambda}_{\nu} - \partial_\nu\partial_\rho h^{\lambda}_{\mu} - \partial_\mu\partial^{\lambda} h_{\nu\rho} + \partial_\nu\partial^{\lambda} h_{\mu\rho}\right) \tag{5.19}$$

I would now like to point out that this expression is actually *gauge-invariant*, i.e. if we do the transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu$, then $R^{\lambda}{}_{\rho\mu\nu}$ does not change.

Let us take a second to verify this: the change in the Riemann tensor is

$$\delta_\zeta R^{\lambda}{}_{\rho\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\rho\left(\boxed{\partial^\lambda\zeta_\nu} + \partial_\nu\zeta^\lambda\right) - \partial_\nu\partial_\rho\left(\partial^\lambda\zeta_\mu + \partial_\mu\zeta^\lambda\right) - \partial_\mu\partial^\lambda\left(\partial_\nu\zeta_\rho + \boxed{\partial_\rho\zeta_\nu}\right) + \partial_\nu\partial^\lambda\left(\partial_\mu\zeta_\rho + \partial_\rho\zeta_\mu\right)\right) \tag{5.20}$$

Now as it turns out, all of these terms cancel pairwise. The easiest way to see this is to pick one of them – say the ζ_ν with the free ν index, and look at the two terms where it appears; by using the commutativity of partials, we see that they are the same but with the opposite sign. Thus they cancel, and so on for every term. We conclude that

$$\delta_\zeta R^{\lambda}{}_{\rho\mu\nu} = 0 \tag{5.21}$$

i.e. the *Riemann curvature tensor is gauge-invariant.* Physically, what this means is that the notion of curvature does not care about what what coordinate system you use to describe it. At a practical level, this establishes what I said earlier; if h is a solution to Einstein's equation (which depends only on $R^{\lambda}{}_{\rho\mu\nu}$), then so is its gauge-transformation.

Moving on, we contract on λ and μ in the expression for (5.19) to get the linearized Ricci tensor:

$$R^{\lambda}{}_{\rho\lambda\nu} = \boxed{R_{\rho\nu} = \frac{1}{2}\left(\partial_\mu\partial_\rho h^{\mu}_{\nu} - \partial_\nu\partial_\rho h^{\lambda}_{\lambda} - \partial_\mu\partial^{\mu} h_{\nu\rho} + \partial_\nu\partial^{\mu} h_{\mu\rho}\right)} \tag{5.22}$$

If we set this to zero, then we obtain the equation that we need to solve for h. It is a linear equation.

5.2.2 Solving linearized wave equation

Now, I want to note that we can actually simplify this a lot by using the gauge transformations that we have discussed at such length. In particular, I would like to use this freedom:

$$h'_{\mu\nu} = h_{\mu\nu} + \delta_\zeta h_{\mu\nu} \tag{5.23}$$

to pick an h' so that h' satisfies the following *harmonic gauge condition.*

$$\partial_\mu h'^{\mu}_{\nu} - \frac{1}{2}\partial_\nu\left(h'^{\lambda}_{\lambda}\right) = 0 \tag{5.24}$$

The reason for this is that it simplifies greatly the equations of motion. It turns out we can always find a ζ that does the job: plugging in (5.23) we see that

$$\partial_\mu h'^{\mu}_{\nu} - \frac{1}{2}\partial_\nu\left(h'^{\lambda}_{\lambda}\right) = \partial_\mu h^{\mu}_{\nu} - \frac{1}{2}\partial_\nu\left(h^{\lambda}_{\lambda}\right) - \partial_\mu\partial^\mu\zeta_\nu \tag{5.25}$$

So if we just pick ζ_ν to satisfy:

$$\partial_\mu\partial^\mu\zeta_\nu = \partial_\mu h^{\mu}_{\nu} - \frac{1}{2}\partial_\nu\left(h^{\lambda}_{\lambda}\right) \tag{5.26}$$

then h' will indeed satisfy the harmonic gauge condition. View this as a PDE for ζ_ν; it will always *have* a solution. Note that ζ_μ does not have a *unique* solution: we can add to it any ζ_μ that satisfies $\partial_\mu \partial^\mu \zeta_\nu = 0$. This is called a *residual gauge transformation.*

However, we can now drop the prime, and simply assume that we started all along with an h that satisfied:

$$\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu \left(h^\lambda_\lambda\right) = 0 \tag{5.27}$$

The point of this is to kill many terms from (5.22):

$$R_{\rho\nu} = -\frac{1}{2}\partial_\mu \partial^\mu h_{\nu\rho} \tag{5.28}$$

Setting this to 0 for a gravitational wave propagating through the vacuum, we find at the end of the day:

$$\boxed{\partial_\mu \partial^\mu h_{\nu\rho} = 0} \tag{5.29}$$

This is the *relativistic wave equation*: this is the whole point of the analysis, from here we may now see that indeed general relativity admits gravitational wave solutions. Let us take a moment to understand what the operator $\partial_\mu \partial^\mu$ does: expanding it out, we see that it looks like

$$\left(\partial_t^2 - \delta^{ij}\partial_i \partial_j\right) h_{\mu\nu} = 0 \tag{5.30}$$

You will recall from your earlier classes that solutions to this equation are travelling waves. Note that e.g. if we assume that the wave depends only on t and z then we see that

$$h_{xy} = \phi(t \pm z) \tag{5.31}$$

is a solution, for an arbitrary function ϕ: this represents a wave that is traveling in the z direction, at speed 1 (i.e. the speed of light).

Let's be more systematic about this. We normally *instead* solve this equation in Fourier space by writing out a solution as follows

$$h_{\mu\nu} = \mathrm{Re}\left(\epsilon_{\nu\rho} e^{-ik_\mu x^\mu}\right) \qquad k_\mu = (\omega, k_i) \tag{5.32}$$

with k_μ a four-vector that is called the *momentum of the wave* k_i tells you the direction that the wave is moving, and ω tells you its frequency. $\epsilon_{\mu\nu}$ is a constant tensor that tells us "the directions the wave points in": its called the *polarization tensor*. We see from plugging this ansatz into (5.29) that we have

$$k_\mu k^\mu \epsilon_{\nu\rho} e^{-ik_\mu x^\mu} = 0 \qquad \to \qquad k_\mu k^\mu = 0\ . \tag{5.33}$$

In other words, the momentum is a null vector. This implies that the wave travels at the speed of light.

Let us turn now to the polarization tensor $\epsilon_{\mu\nu}$. It is conventional to make the following further choices on $\epsilon_{\mu\nu}$: we use the *residual gauge transformation* to set the following condition on $\epsilon_{\mu\nu}$:

$$\epsilon_{0i} = 0 \qquad \epsilon_{\mu\nu}\eta^{\mu\nu} = 0 \tag{5.34}$$

This is always possible[5].

Note that now the harmonic gauge condition (5.24) becomes simply.

$$k^\mu \epsilon_{\mu\nu} = 0 \tag{5.35}$$

[5]We do not discuss this in lecture, but note that any $\zeta = b_\mu e^{-ik\cdot x}$ with $k^2 = 0$ satisfies the residual gauge condition, and now b and k can be fixed to satisfy these conditions

This collection of conditions is called *tranvserse traceless gauge*.

Let's write down concretely what these conditions mean; consider a wave that is propagating only in the z direction. Then if we sort through all the conditions we have for $\epsilon_{\mu\nu}$, we eventually find that it is:

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_+ & \epsilon_\times & 0 \\ 0 & \epsilon_\times & -\epsilon_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.36}$$

Here ϵ_+ and $\epsilon_\times$ are two undetermined parameters: they represent the two *polarizations of a gravitational wave* that is moving in the z direction. More physically, they represent the ways that spacetime wiggles if you hit it.

Note: there are two of them precisely because we said that there are 2 degrees of freedom in the gravitational field. This is the same information.

Now what does this wave *do*? It distorts spacetime by changing physical distances. Let us think about this for a second: consider a single particle that is initially at rest at a point, so

$$U^\mu = (1, 0, 0, 0) \tag{5.37}$$

Now suppose a gravitational wave comes along. We know that the particle will follow the geodesic equation, which tells us that

$$\frac{d^2}{ds^2} U^\mu + \Gamma^\mu_{\rho\sigma} U^\rho U^\sigma = 0 \tag{5.38}$$

Now since the particle is *initially* at rest, the instantaneous value of its acceleration at $t = 0$ is

$$\frac{d^2}{ds^2} U^\mu \Big|_{t=0} = -\Gamma^\mu_{tt} \tag{5.39}$$

But if we now turn back to (5.17) we see that

$$\Gamma^\mu_{tt} = \frac{1}{2} \eta^{\mu\nu} \left(2\partial_t h_{t\nu} - \partial_\nu h_{tt} \right) = 0 \tag{5.40}$$

In other words, the acceleration is zero! So it seems like the particle does not move as the gravitational wave passes by? Is this correct?

No. What we have actually shown is that the coordinate values of the particle do not change; but this is not a physically meaningful statement (coordinates are labels, etc. etc.). A physically meaningful statement would be something like, "what is the proper distance between two particles", etc. etc. So let us calculate that: consider two particles separated in coordinates by

$$\xi^\mu = (0, \xi^x, \xi^y, 0) \tag{5.41}$$

As we have seen, if the particles are at rest, their coordinates do not more, and so this defines the separation for all time. But now we can calculate the proper distance between these two particles to find:

$$\Delta = g_{\mu\nu} \xi^\mu \xi^\nu = \left(\eta_{\mu\nu} + h_{\mu\nu} \right) \xi^\mu \xi^\nu = -\delta_{ij} \xi^i \xi^j + \boxed{\left(2\epsilon_\times \xi^x \xi^y + \epsilon_+ \left((\xi^x)^2 - (\xi^y)^2 \right) \right) \cos(k \cdot x)} \tag{5.42}$$

The proper distance changes! Thus the particles clearly wiggle. If you think about what this is doing, consider arranging a bunch of particles in a circle; as a gravitational wave passes by, depending on the polarization of the wave the circle will distort in the manner shown in Figure.

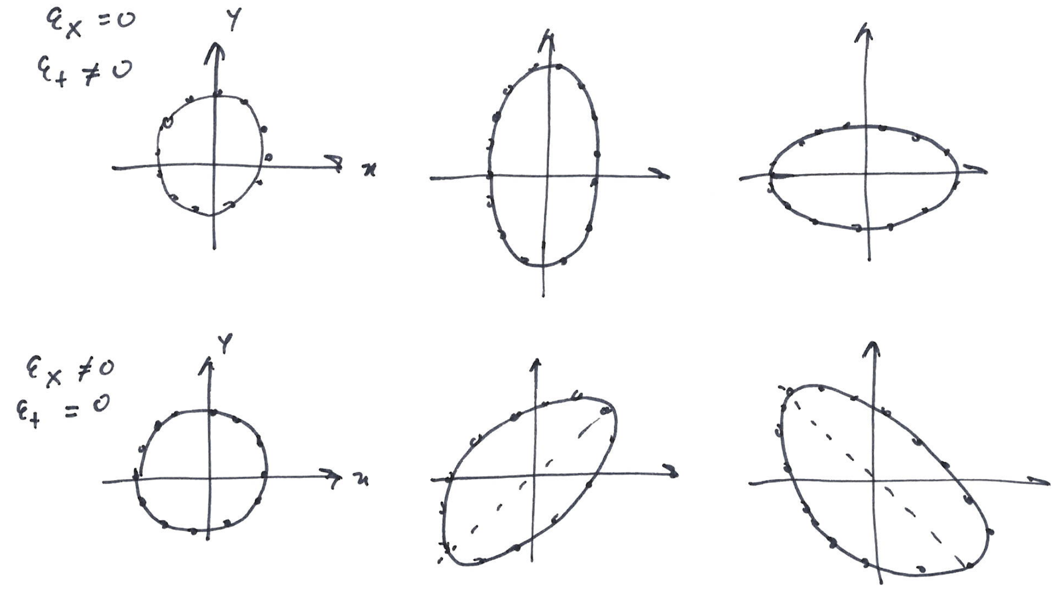

Figure 5.2: Effect of two possible gravitational wave polarizations on test particles arranged in a circle in the xy plane.

5.3 Production of gravitational waves

Here I will be very sketchy, as this is an intricate subject. The basic idea is that up till now we have linearized Einstein's equations in *vacuum*, and found that the resulting perturbations obey the wave equation. What one does next is instead linearize them in the presence of matter, i.e. a non-trivial $T_{\mu\nu}$;

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{5.43}$$

Going through the linearization one finds

$$\partial_\mu \partial^\mu h_{\nu\rho} = -16\pi G \left(T_{\nu\rho} - \frac{T}{2} g_{\nu\rho} \right) \tag{5.44}$$

The point here is that non-trivial matter acts as a source for gravitational waves. So almost any kind of non-trivial matter will do it; however to create a radiation field it has to be *moving* in some sense. Not only that, it has to move in a non-spherically symmetric way: this is due to Birkhoff's theorem, which states that

The spacetime outside of any spherically symmetric (possibly time-dependent) source is the Schwarzschild metric.

In other words, if you take a balloon and inflate and deflate it in a spherically symmetric way, it will not create any gravitational radiation, because the spacetime outside remains Schwarzschild always. On the other hand, if you take two stars and spin them around each other, then this *will* create gravitational radiation (though I have not shown this. See references in lecture notes.)

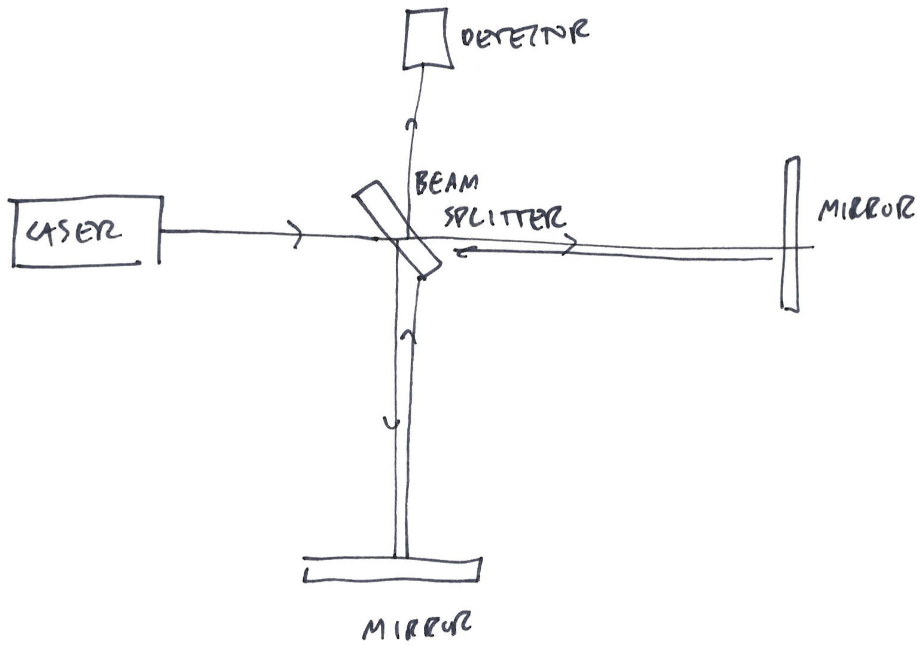

Figure 5.3: Schematic picture of laser interferometer

5.4 Detection of gravitational waves

Until recently, the only detection of gravitational waves was indirect and came from the Hulse-Taylor binary pulsar. This is a system of two neutron stars (21 thousand light years away) that orbit each other once every seven hours. As time goes on, this system creates gravitational waves that radiate away some of its energy: thus they come closer to each other slowly, and the orbital period decreases. We can measure the orbital period by observing the (electromagnetic) radiation produced by the star on Earth. People have been watching it for years and years, and the decrease of the orbital period is perfectly consistent with the prediction for the energy carried away by gravitational waves in general relativity.

However, all of this recently changed. To understand what happened, we first have to consider how *direct* detection of gravitational waves would work. This is done through an interferometer: you evacuate a 4 km tunnel on earth from air and set up a system of laser beams and mirrors. One then interferes the beams together and waits. If a gravitational waves passes by, it will squeeze the tunnels and alter the interference pattern; however the signal is truly miniscule! The signal that we are looking for has a relative strength of 10^{-21}; over a 4 km tunnel, this alters the mirror location by about 10^{-18} m. By comparison, a proton has a radius of about 10^{-15} m: so the signal is about a thousandth the size of a single proton. It is amazing that we can detect it.

But we can! About a billion years ago, two black holes collided and made a bigger one. They created a spectacular amount of radiation, emitting about 5% of their total rest mass into gravitational radiation, which spread out and hit the earth on September 14, 2015. More precisely, it first hit the detector in Lousiana; 7 ms later, it propagated through the earth and hit the detector in Washington state. The signal created set off all kinds of alarms and is spectacularly good agreement with the predictions of general relativity **Handout**. Since then, there have been a few more events and we expect many many more to come.

This is thus the dawning of a new age of astronomy; previously we would only *look* out into the sky, but now we can close our eyes and listen to what the ripples in spacetime are telling us. It remains to be seen what we will find.

6 Advanced topics: Penrose Diagrams

Here we discuss one more topic; you will have noticed that a lot of the interesting stuff in this course involved understanding the *causal structure* of spacetime; e.g. Romeo receives no messages from Juliet once she passes the horizon, etc. etc. We will now learn how to draw pictures that let you draw a picture of an entire universe on a (finite) piece of paper in a way that captures the *causal* structure; these are called *Penrose diagrams*.

It is best to explain this with an example. Consider ordinary Minkowski space in polar coordinates:

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2 \tag{6.1}$$

We will need to be careful about the range of the coordinates: we have

$$t \in (-\infty, +\infty) \qquad r \in [0, \infty) \tag{6.2}$$

In this section we will discuss only radial light rays; as we have discussed extensively, these follow $t = \pm r$. Note that it is a bit silly to try to draw a picture of the spacetime; as the coordinates (t, r) go off to infinity.

To fit the spacetime on a finite blackboard, we need to change to coordinates that have a finite range, and we need to do this in a way that preserves the causal structure. To this end, first consider the following change of coordinates:

$$u \equiv t - r \qquad v \equiv t + t \tag{6.3}$$

These sorts of coordinates are often called a "light-cone" coordinate system, because light rays satisfy $u = \text{const}, v = \text{const}$. What is the range? Note that we have

$$u \in (-\infty, +\infty) \qquad v \in (-\infty, +\infty) \qquad u \leq v \tag{6.4}$$

The last relation is crucial, and follows from the fact that $r > 0$ before. Now we can write the metric in terms of u, v; we have $r = \frac{1}{2}(v - u)$, and thus

$$ds^2 = dudv - \frac{1}{4}(v - u)^2 d\Omega^2 \tag{6.5}$$

Now is the magic part: we now squish the entire infinite range of u, v into a finite range of a new coordinate U, V using the arctan function:

$$U \equiv \arctan u \qquad V \equiv \arctan v \tag{6.6}$$

which of course means that

$$u = \tan U \qquad v = \tan V \tag{6.7}$$

Recall the shape of the arctan function; note that as its argument goes between $-\infty$ and $+\infty$, the arctan itself only goes between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Thus we have for the ranges of the coordinates:

$$U \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \qquad V \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \qquad U \leq V \tag{6.8}$$

The last property is inherited from above, as the arctan function is monotonic. We can now convert the metric to the new coordinates. We use $du = \frac{1}{\cos^2 U} dU$ (and similarly for v). We also note that

$$(v - u)^2 = (\tan V - \tan U)^2 = \frac{1}{\cos^2 U \cos^2 V} \sin^2(V - U) \tag{6.9}$$

where the last equality requires some trig identities (check at home!). Putting it together, we find that the metric takes the form:

$$ds^2 = \frac{1}{\cos^2 U \cos^2 V}\left(dU dV - \frac{1}{4}\sin^2(V-U)d\Omega^2\right) \tag{6.10}$$

This is starting to look good; you will see why in a second. For purposes of intuition, we now switch back to a timelike coordinate T and a spacelike coordinate R:

$$T \equiv V + U \qquad R \equiv V - U \tag{6.11}$$

which results in the following ranges: the one for R is simply:

$$\boxed{R \in [0, \pi)} \tag{6.12}$$

and we can write the range for T in the following way:

$$T + R = 2V < \pi \qquad -T + R = -2U < \pi \tag{6.13}$$

which combines into the following handy relation:

$$\boxed{|T| + R < \pi} \tag{6.14}$$

We can now write the metric as follows:

$$ds^2 = \frac{1}{\cos^2 U \cos^2 V}\left(dT^2 - dR^2 - \frac{1}{4}\sin^2 R d\Omega^2\right) \tag{6.15}$$

The point of all of this analysis was actually to figure out the *ranges* on the coordinates. In particular, note that light rays in the new coordinates (T, R) follow:

$$T = \pm R \tag{6.16}$$

and now we can draw a simple picture on the (T, R) plane: we get simply **picture**.

This is called the Penrose diagram of Minkowski space; it shows you the entire spacetime in one easy-to-draw picture.

A Christoffel symbols

Here I will document some useful Christoffel symbols.

A.1 Friedmann-Robertson-Walker metric

The metric is

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right) \tag{A.1}$$

and the Christoffel symbols are

$$\Gamma^t_{rr} = \frac{a\dot{a}}{1 - \kappa r^2} \tag{A.2}$$
$$\Gamma^t_{\theta\theta} = r^2 a\dot{a} \tag{A.3}$$
$$\Gamma^t_{\phi\phi} = r^2 a\dot{a} \sin^2\theta \tag{A.4}$$
$$\Gamma^r_{rt} = \frac{\dot{a}}{a} \tag{A.5}$$
$$\Gamma^r_{rr} = \frac{r\kappa}{1 - \kappa r^2} \tag{A.6}$$
$$\Gamma^r_{\theta\theta} = r(\kappa r^2 - 1) \tag{A.7}$$
$$\Gamma^r_{\phi\phi} = r(\kappa r^2 - 1)\sin^2\theta \tag{A.8}$$
$$\Gamma^\theta_{\theta t} = \frac{\dot{a}}{a} \tag{A.9}$$
$$\Gamma^\theta_{\theta r} = \frac{1}{r} \tag{A.10}$$
$$\Gamma^\theta_{\phi\phi} = -\cos\theta \sin^\theta \tag{A.11}$$
$$\Gamma^\phi_{\phi t} = \frac{\dot{a}}{a} \tag{A.12}$$
$$\Gamma^\phi_{\phi r} = \frac{1}{r} \tag{A.13}$$
$$\Gamma^\phi_{\phi\theta} = \cot\theta \tag{A.14}$$

B Propagating degrees of freedom

Here is some bonus (unexamined) material for those who want an abstract understanding of how many degrees of freedom there are in gravity.

B.1 Propagating degrees of freedom: trivial example

First, I want to explain what a propagating degree of freedom *is*.

This may seem like an intuitive concept, but let us slowly build up to the full case. In the full case, we will have dependence on space and time both; it turns out that the spatial part just goes along for the ride, so let us first work out an example without it. We then start with on ODE, e.g. the following for a single variable $X(t)$:

$$\ddot{X}(t) + \omega^2 X(t) = 0 \qquad X = x_1 e^{i\omega t} + x_2 e^{-i\omega t} \tag{B.1}$$

So, definition: a propagating degree of freedom is one that obeys a 2nd order differential equation. Thus, here we have one propagating degree of freedom; it requires two integration constants to determine a full solution.

Now consider the following system $(X(t), Y(t))$, and suppose it satisfies the equations:

$$\ddot{X} + \omega^2 X = 0 \qquad \ddot{Y} + \omega^2 Y = 0 \tag{B.2}$$

whose general solution is

$$X = x_1 e^{i\omega t} + x_2 e^{-i\omega t} \qquad Y = y_1 e^{i\omega t} + y_2 e^{-i\omega t} \tag{B.3}$$

Clearly we have two propagating degrees of freedom; we need four integration constants to specify a full solution. We may think that we specify this data at $t = 0$; these integration constants are then called "initial data". We will call equations with two time derivatives "dynamical equations".

Now let us add a wrinkle: consider adding another equation, e.g.

$$\dot{X} + \dot{Y} = 0 \tag{B.4}$$

This imposes a relation between the different integration constants: we see from above that it tells us that

$$x_1 = -y_1 \qquad x_2 = -y_2 \tag{B.5}$$

Thus this is a *constraint on the initial data*: it tells us that not all sorts of initial data are possible. The constraint brings us down to having one degree of freedom again. Such constraints will appear whenever we have an equation that has fewer time derivatives than the dynamical equation.

B.2 Propagating degrees of freedom: general relativity

Now, with this worked out, we turn to the Einstein equations in vacuum:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{B.6}$$

Remember that $R \sim \partial\Gamma$ and $\Gamma \sim \partial g$; thus these are second order equations in space and time. We want to find out how many propagating degrees of freedom there are in general relativity in four dimensions. It turns out that some of the equations are dynamical ones, and some are constraint equations: however they are too complicated to actually write down, so we need to be more clever. Today we will give an indirect argument that works at the full nonlinear level; tomorrow we will perturb around flat space and work out the analog of directly finding the integration constants.

Naively, the metric has 10 components. So if there were no wrinkles, we would expect 10 dynamical degrees of freedom. But some of the Einstein equations are constraints; to find out which they are, we need to look inside $G_{\mu\nu}$ and see which equations have fewer than 2 time derivatives; from the analysis above, these will be the constraints.

Now recall from Michaelmas term the contracted Bianchi identity:

$$\nabla_\mu G^{\mu\nu} = 0 \tag{B.7}$$

Remember that this works for *any metric*, not just for a solution to the Einstein equation[6]. Now let's expand this out using Christoffels, etc:

$$\nabla_\mu G^{\mu\nu} = \partial_\mu G^{\mu\nu} + \Gamma^\mu_{\mu\sigma} G^{\sigma\nu} + \Gamma^\nu_{\mu\sigma} G^{\mu\sigma} = 0 \tag{B.8}$$

[6]Indeed, it is clearly trivial for a solution to the Einstein equations.

Now I break this into space and time and rearrange a little bit:

$$\partial_t G^{t\nu} = -\left(\partial_i G^{i\nu} + \Gamma^{\mu}_{\mu\sigma} G^{\sigma\nu} + \Gamma^{\nu}_{\mu\sigma} G^{\mu\sigma}\right) \tag{B.9}$$

Now: recall that this works for *any metric* g. Suppose $G^{t\nu}$ had a term in it like two time derivatives of $g_{\alpha\beta}$: $G^{t\nu} \supset \partial_t^2 g_{\alpha\beta}$. Then the left hand side of this equation would have a term with *three* time derivatives in it. But we know that the right hand side has at most two *derivatives*, of any kind. Therefore there cannot be any term in $G^{t\nu}$ with two time derivatives in it. Thus each of the $G^{t\nu}$ equations is a constraint.

So we have four constraints. Now we are down to $10 - 4 = 6$ degrees of freedom.

Next, note that we also have four gauge transformations; thus there are 4 functions worth of freedom that are arbitrary, and we can remove those two. So now we have

$$10 - 4 - 4 = 2 \tag{B.10}$$

So there are 2 propagating degrees of freedom left in general relativity in four dimensions. This means that there are 2 polarizations of gravitational wave that we need to look for. In the next section, we will explicitly look for them.